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Stability and Vibration of Loaded Structural Members

Synopsis of PhD theses by

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1 Preliminaries & objectives

Structural elements such as beams are integral parts in numerous engineering applications, including buildings, bridges, aerospace structures, and marine frameworks. Their stability and vibrational behavior are pivotal to ensuring the safety and performance of these systems.

Since Euler's contributions, substantial progress has been made in the understanding of buckling. Extensive research has been dedicated to the stability of various structural elements, including shells, columns, arches, and other load-bearing components [1, 2, 3]. Focusing specifically on column buckling, the book [4] offers a comprehensive analysis of the theory of elastic stability for columns subjected to continuous axial loading. The study presented in [5] examines the static and dynamic stability of columns under self-weight using experimental, analytical, and numerical approaches.

The concept of the Green function first emerged in Green's 1828 publication [6], where he introduced the Green theorem and applied the Green function to electrostatic problems governed by partial differential equations. Bocher extended the definition of the Green function to two-point boundary value problems (BVPs) governed by ordinary differential equations in [7]. Green function concept is systematically published in [8], providing a foundational understanding of its applications. Early works [9,10] further define and analyze the properties of the Green function for two-point BVPs involving ordinary linear differential equations, offering a comprehensive collection of Green functions in closed form for various conditions. Regarding degenerate ordinary differential equation systems, new findings are presented in [11]. For certain three-point BVPs governed by second-order linear ordinary differential equations, the corresponding Green's functions are comprehensively discussed in Zhao's work [12]. Additionally, multi-point BVPs are extensively studied in the literature. Notably, references [13, 14, 15] investigate and solve specific four-point BVPs, including singular fourth-order p-Laplacian equations, nonlinear BVPs utilizing fixed points of strict set-contractions, and applications of the Leray-Schauder theorem. Furthermore, the study presented in [16] proposes a numerical iterative algorithm for solving nonlinear four-point BVPs.

The study of beam vibrations has been a significant area of research, with numerous scientific contributions in the literature. For instance, the work presented in [17] investigates the transverse vibrations of buckled beams by formulating partial differential equations that incorporate nonlinear mid-plane strain effects. These equations are subsequently transformed into ordinary differential equations through the application of the Galerkin method. The free vibrations of beams resting on an elastic Pasternak foundation with spatially varying material properties are examined in [18]. The equations of motion are derived using Hamilton's principle, and the influence of various parameters, including geometry, material properties, and foundation stiffness, is analyzed.

Additionally, the study [19] investigates the free vibration behavior of composite beams incorporating higher-order shear deformation theory, employing the isogeometric collocation method. A key advantage of this technique is its efficiency, as it requires only a single integration point per element. The study [20] investigates the free vibrations of stepped beams with intermediate elastic connections. Additionally, paper [21] addresses the vibrations of multi-stepped beams with multiple concentrated elements using the continuous mass transfer matrix method, providing closed-form solutions for the free vibration frequencies of Timoshenko beams. Furthermore, paper [22] introduces the Lumped Mass Transfer Matrix Method to analyze the free vibrations of stepped axially functionally graded beams with point masses. This method is both powerful and relatively simple, offering efficient solutions for this class of problems. The work [23] examines systems composed of multiple beams, focusing on transverse free and forced vibrations while assuming viscoelastic connections between the elements. Additionally, research [24] investigates self-weight-loaded columns and cables, providing both analytical and numerical solutions using finite element software. The study [25] presents a thermoelastic dynamic analysis of micro- and nanobeams. The coupled equations are solved using the Green's function method, making the model suitable for analyzing free and forced vibrations. The Green functions are obtained through the eigenfunction expansion and Laplace transform methods, with resulting Fredholm integral equation solved using the kernel function method. Additionally, article [26] investigates the dynamic response of damped Timoshenko beams subjected to bending and torsion. The Green functions for arbitrary boundary conditions are derived using the Laplace transform method.

Curved beams, widely utilized in modern structural applications, introduce additional complexities due to their geometric characteristics. Their increasing popularity is attributed to their favorable mechanical behavior under compression and their aesthetic appeal in contemporary architecture. These structural elements are extensively employed in aerospace, civil, and marine engineering sectors [27, 28]. Numerous studies have investigated the behavior of curved beams to provide engineers with practical insights into their stability. Classical theories, such as those developed by Simitses and Timoshenko, predict elastic buckling loads and offer approximations for the classical buckling load of sinusoidal shallow arches subjected to evenly distributed loads [29, 30]. Subsequent researchers have expanded these approaches, leading to closed-form solutions and finite element analyses under various assumptions, as seen in [31, 32]. The accurate prediction of buckling loads is crucial for structural resistance design [33]. The effects of concentrated, uniform, and asymmetrically distributed mechanical loads have been examined by analyzing equilibrium paths. For instance, article [34] presents an analytical model based on the virtual work principle, capable of addressing the in-plane elastic stability of shallow parabolic arches supported by horizontal springs and subjected to uniformly distributed loads. Additionally, numerical studies have demonstrated that the position of radial loads significantly influences nonlinear equilibrium and limit-point buckling loads [35, 36]. The in-plane elastic static stability of circular beams with cross-sectional inhomogeneity under vertical loading at the crown point was investigated in [37]. Early research on the impact of geometric imperfections on the stability of shallow arches is documented in [38] and [39]. More recently, paper [40] highlights that slightly imperfect homogeneous shallow arches with interlayer slip may exhibit multiple unconnected remote equilibrium paths. Furthermore, Chroscielewski et al. [41] address the complexities of solving nonlinear BVPs for elastic structures, particularly focusing on the post-buckling behavior of shear-deformable circular arches. Their study discusses the numerical challenges associated with obtaining solutions for these structures, particularly in highly nonlinear regimes.

Based on the literature review, the following objectives and key research challenges are tackled in my thesis work.
Objective 1: The primary objective is to develop a comprehensive stability analysis framework for heterogeneous beams with three supports using the Green's function technique. The details are as follows:
- To tackle the stability problem of heterogeneous beams with three supports, particularly those with intermediate spring supports. The stability problems of these beams are to be given by three-point boundary value problems.
- To clarify the properties of the Green function for the considered three-point eigenvalue problems and provide their calculations.
- To transform the eigenvalue problems established for the critical load into eigenvalue problems governed by homogeneous Fredholm integral equations.
- Solving these integral equations numerically enables the determination of critical loads, offering fresh insights into the stability of such beams.
- To investigate the impact of the middle support position on the ultimate load bearing abilities.

Objective 2: My second objective focuses on extending the three BVPs analysis to a four-point BVPs. The key aims are as follows:
- To elucidate the structure of the Green function specifically tailored for a class of four-point boundary value problems, generalizing results from three-point boundary value problems.
- To utilize the constructed Green function to transform four-point eigenvalue problems into homogeneous Fredholm integral equations, with the Green function serving as the kernel function.
- To calculate the eigenvalues for the free vibrations of the considered beams.
- To develop a solution algorithm for the eigenvalue problems governed by ho-

mogeneous Fredholm integral equations by reducing these integral equations to algebraic eigenvalue problems, which are then solved numerically.
- To provide example about the applicability of the technique, involving heterogeneous beams with four supports.
- To validate the numerical results using commercial Finite Element (FE) software.

Objective 3: Building on the previous objectives, Objective 3 focuses on the application of the Green's function technique to study the vibration and stability of stepped heterogeneous beams. Specifically, the goals of this objective are as follows:
- To develop and formalize the definition of Green functions tailored specifically for coupled boundary value problems. To clarify the intrinsic properties of these Green functions and devise methodologies for calculating their specific elements.
- To demonstrate the application of these Green functions to stepped heterogeneous beams with two supports. To focus on beams fixed and pinned at both endpoints, covering scenarios both with and without axial loads.
- To calculate the eigenfrequencies for both unloaded and axially loaded stepped beams.
- To transform the eigenvalue problems into homogeneous Fredholm integral equations, solve these numerically, and provide accurate eigenfrequency data.
- To assess the impact of axial tensile or compressive loads by replacing classical eigenvalue problems with Fredholm integral equations.
- To provide detailed analysis and numerical solutions to the stability problems.
- To validate the numerical solutions by comparing them with results obtained by FE software.

Objective 4: In my objective 4, I focus on incorporating geometrical imperfections into a one-dimensional arch model to study the in-plane static stability of fixed arches subjected to a radial concentrated loads. The specific aims of this investigation are as follows:
- To assemble a geometrically non-linear mechanical model that accounts for initial shape error in order to assess the arch sensitivity to this kind of imperfection.
- To derive the equilibrium equations from a variational principle.
- To solve these equations analytically.
- To identify the limit points on the non-linear equilibrium path and analyze the effects of geometric parameters on the critical load.
- To extend the analysis by including various arch geometries to provide a comprehensive understanding of their stability behavior.

2 Investigations performed

In this section, the following notations are introduced for clarity:

- χ: Spring stiffness parameter, representing the stiffness of the intermediate spring support – $\chi = \frac{kL^3}{I_{ey}}$,
- k: Spring stiffness,
- L : Length of the beam,
- I_{ey} : E-weighted moment of inertia with respect to the axis y,
- FrrF and PrrP: Designations for fixed-fixed and pinned-pinned beams with two roller supports, respectively,
- $\hat{b}$ and $\hat{c}$: Positions of intermediate supports,
- b and c: Dimensionless positions of intermediate supports – $b = \hat{b} \backslash L$; $c = \hat{c} \backslash L$,
- λ_i: The i-th eigenvalue,
- $\omega_{1 \text{ no load}}$: The first natural circular frequency of unloaded beams,
- $\omega_{1c}; \omega_{1t}$: The first circular frequency for compression; tension,
- N: Axial load,
- $\mathcal{N}$: Dimensionless load – $\mathcal{N} = \frac{NL^2}{I_{ey}}$,
- PPStp: Abbreviation for pinned-pinned stepped beam,
- α and κ: Dimensionless parameters; $\alpha = \dfrac{I_{ey_2}}{I_{ey_1}}$ and $\kappa = \dfrac{\rho_{a2} A_2 I_{ey_1}}{\rho_{a1} A_1 I_{ey_2}}$ with A - cross-sectional area and ρ_a - average density,
- φ: Angle coordinate,
- $H(\varphi)$: Arbitrary geometrical imperfection through initial radial displacements,
- a and n: The dimensionless amplitude of the imperfection and the mode number,
- θ: Semi-vertex angle of the arch.

In my thesis, I focused on the stability and vibration of straight beams and arches with specific boundary conditions. One of the key aspects of my investigation involved the stability of beams with three supports, where the middle support is a spring that restrains vertical motion. I assumed cross-sectional heterogeneity, based on the approach introduced in [42]. I used an approach from [43] to derive Green functions for beams with fixed-fixed, fixed-pinned, and pinned-pinned boundary conditions that include an intermediate linear spring support (referred to as FssF, FssP, and PssP beams, respectively. The introduction of a spring stiffness parameter, denoted by χ allowed the derivation of these Green functions for various configurations. As $\chi \to \infty$, these Green functions simplified to those for beams with intermediate roller supports. While for $\chi = 0$, they are reduced to classical boundary conditions (fixed-fixed, fixed-pinned and pinned-pinned beams). Figure 1 presents the dimensionless critical force, $\sqrt{\mathcal{N}_{\text{crit}}}/\pi$, for an FssF beam across various values of b and χ. The graph indicates that when $\chi = 0$, meaning no intermediate support is present, the beam behaves like a conventional fixed-fixed beam, maintaining a constant dimensionless critical force of 2.0. As χ increases, representing a stiffer intermediate support, the critical force rises steadily and significantly, reaching its peak at $b = 0.5$. For sufficiently large values of χ, the beam effectively behaves as if it had an almost rigid support at the midpoint, greatly enhancing its load-bearing capacity.

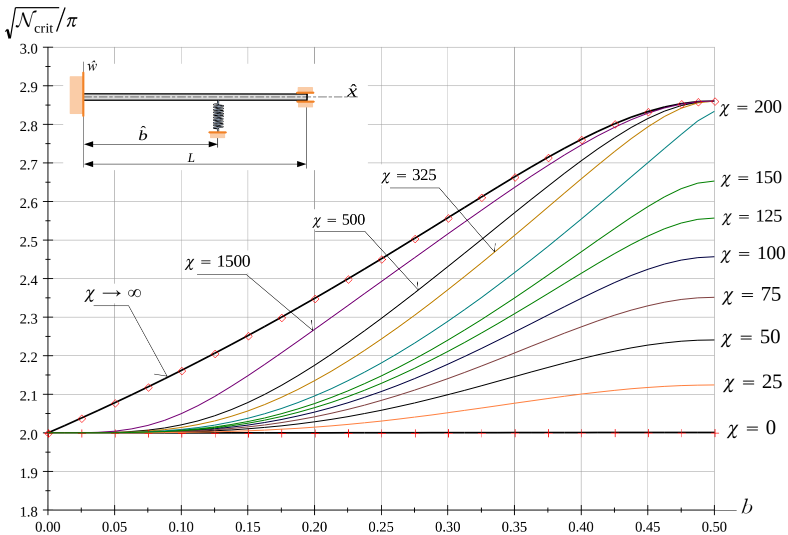

Figure 1. The dimensionless critical force for an FssF beam.

Figure 2 illustrates the dimensionless critical force for FssP beams as a function of b. The maximum resistance to buckling occurs at approximately

$b = 0.63$. For $\chi = 0$, the results correspond to a standard fixed-pinned beam, yielding a constant critical force of 1.4303. As χ approaches infinity, the beam behaves as if it has a rigid intermediate support, with the critical force reaching a peak of approximately 2.458.

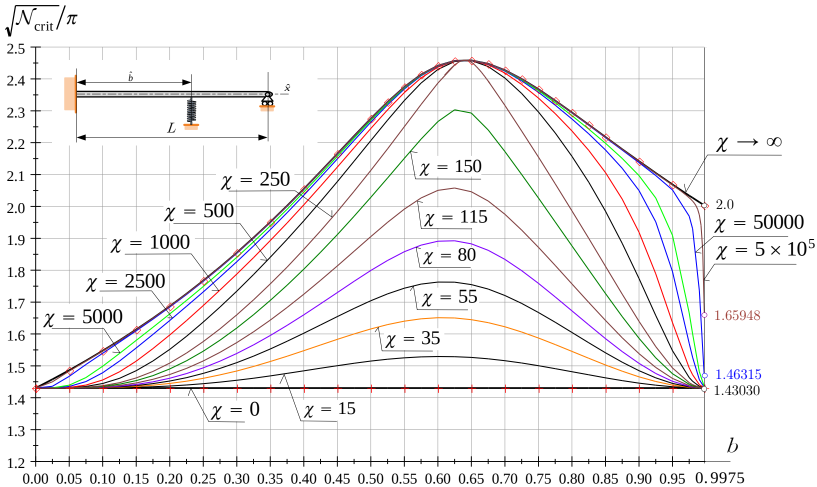

Figure 2. The dimensionless critical force for an FssP beam.

The stability analysis was formulated as an eigenvalue problem governed by a homogeneous Fredholm integral equation, which was solved numerically using the boundary element method. Additionally, the accuracy of the model was validated through comparisons with characteristic equations and FEM, confirming the reliability of the approach for practical engineering applications. The study revealed that increasing the spring stiffness parameter χ and optimizing the support positions could significantly enhance the structural stability of beams. Furthermore, cross-sectional heterogeneity was found to play a crucial role in determining the critical load of the beam. The stability problem results were published in [44].

Furthermore, I examined the vibration behavior of beams using the Green function method, considering four-point boundary value problems. I developed a Green function specifically for four-point BVPs, which allowed for direct computation of the solutions. A unique aspect of my approach was the constructive nature of the Green function, which facilitated the detailed understanding and calculation of the problem. By using this Green function as the kernel, I transformed the eigenvalue problem into a Fredholm integral equation. The analysis focused on the effect of intermediate roller supports positions on the vibration characteristics of FrrF and PrrP beams. I also analyzed limit cases based on support positions, such as $b = 0$, $c = 1$, $c = b + 0.00001$, and found that my results aligned well with results published in [45] and [46], validating the accuracy of my approach in these extreme configurations. Figure 3 shows the function $\sqrt{\lambda_1}/4.73004^2$ against c, with b as parameters. When b is zero, the beam behaves as if it were fixed-fixed, with one intermediate roller support at c. When $c = b + 0.0001$, the two intermediate roller supports are very close, preventing rigid body rotation.

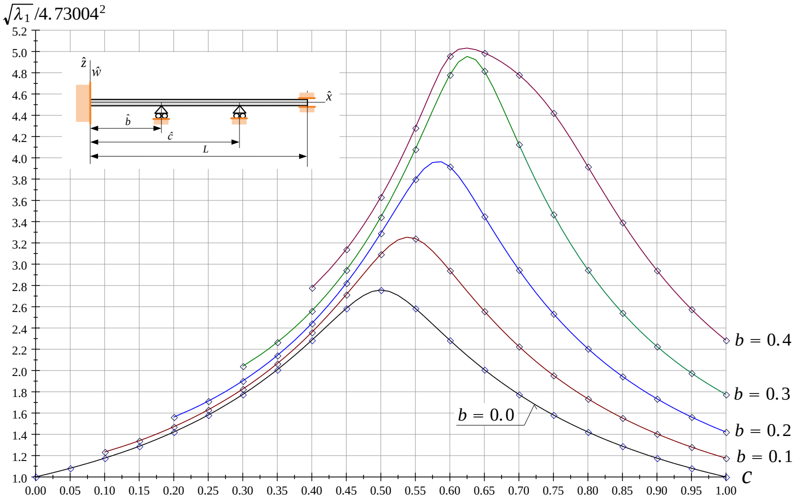

Figure 3. The first eigenvalue against c for FrrF beams.

To further ensure the accuracy of the method, I validated the analytical results with finite element computations, which confirmed the robustness of the approach. This work part was published in [47].

Moreover, I extended my work to stepped beams, where I defined a new class of Green functions tailored for coupled boundary value problems. This enabled direct computation of the Green functions for stepped beams under different boundary conditions. I investigated the properties of these Green functions for stepped beams with fixed-fixed and pinned-pinned boundary conditions under three specific scenarios: : (a) with no axial load, (b) under a compressive force, and (c) under a tensile force. I transformed the eigenvalue problem associated with the free vibrations of these stepped beams into a formulation governed by homogeneous Fredholm integral equations. This approach provides a new method for tackling free vibration problems for stepped beams. The related eigenvalue problem is solved numerically and Figure 4 shows the results for $\sqrt{\lambda_1}/\pi^2$ as function of parameter b. The beam is assumed to have a circular cross-section, with the step location defined by the parameter b.

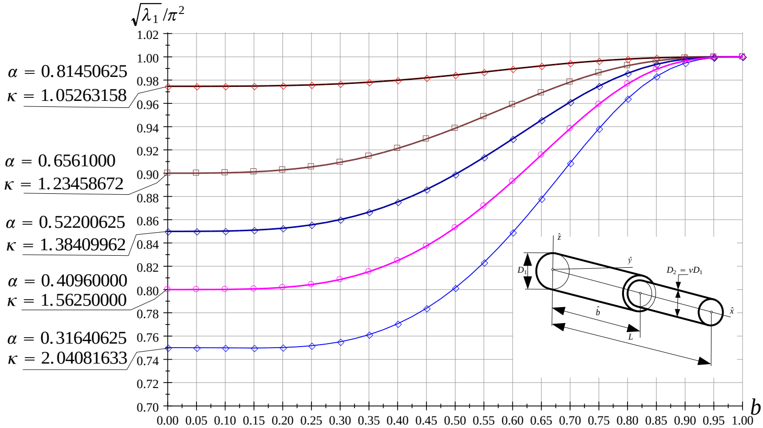

Figure 4. The first eigenvalue of PPStp beam as a function of b; α and κ.

Figure 5 depicts the effect of the axial load on the transverse vibrational frequencies. The quotients $\omega_{1c}^2/\omega_{1\text{ no load}}^2$ and $\omega_{1t}^2/\omega_{1\text{ no load}}^2$ are almost linear functions in $\mathcal{N}/\mathcal{N}_{crit}$.

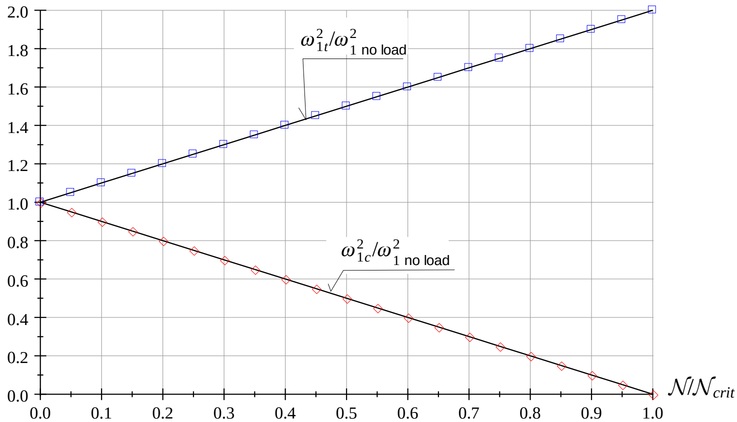

Figure 5. The effect of axial load on the eigenfrequencies when b=0.2

I also validated the results against FEM solutions. The findings of this section of the study were reported in [48].

Finally, I analyzed the stability of arches with geometric imperfections using a geometrically nonlinear model. The model accounted for a linearly elastic material distribution, either homogeneous or functionally graded along the cross-sectional height. A mechanical load was applied at the crown point, and initial displacements inducing strain were considered. Stability was examined and detailed for shallow arches with imperfection functions, such as $H = a\cos(n\frac{\pi\varphi}{2\theta})$ and $H = a\sin(n\frac{\pi\varphi}{\theta})$. The study revealed that for cosine-shaped imperfections, the sign of the imperfection magnitude significantly influenced buckling behavior, whereas sine-shaped imperfections remained unaffected by sign changes. As the arch angle increased, buckling loads approached those of the perfect case, though lower-angled arches were more sensitive to imperfections. Figure 6 presents the variation of the critical load with respect to θ. For $n = 3$, the results indicate an inverse relationship between the critical load and the included angle for negative amplitudes. Additionally, as θ approaches 1, the difference in critical load for a given parameter a becomes more pronounced compared to the case of $n = 1$ for the same geometry.

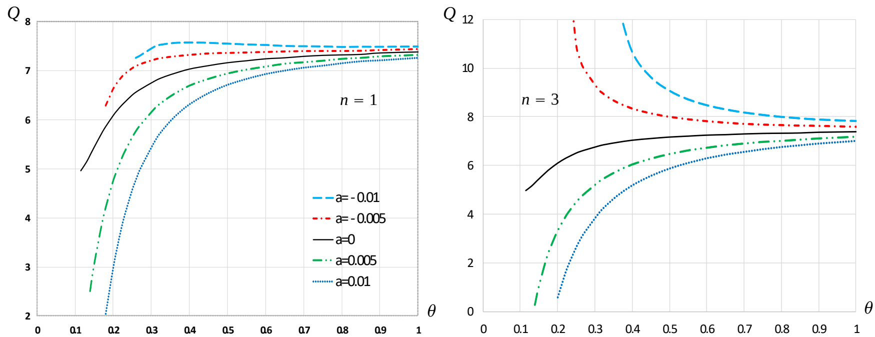

Figure 6. Buckling load against θ for cosine imperfection.

For a sine-shaped imperfection with $n = 2$, Figure 7 exhibits a similar trend to the case of $n = 1$, although with different specific values. At lower angles, imperfections can enhance structural stiffness. Overall, the relative differences between the $n = 1$ and $n = 2$ modes remain minimal, especially when compared to the more pronounced variations observed in the lowest cosine patterns.

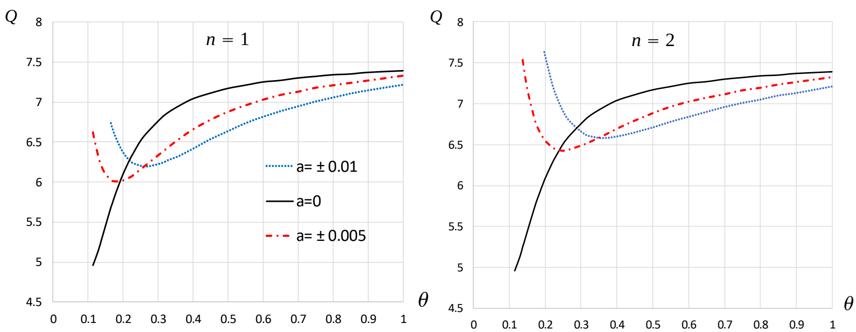

Figure 7. Buckling load against arch semi-vertex angle θ.

It is found that the equilibrium path exhibits two distinct behaviors. When snap-through buckling is not possible, a single stable equilibrium branch exists. However, when a buckling is possible, the equilibrium path consists of three branches: a primary stable branch, a remote stable branch, and an intermediate unstable branch. This portion of the research was presented in [49].

3 Novel results

Thesis 1.

I have investigated the stability problem of straight compressed beams with three supports. The intermediate support is a linear spring, restraining transverse motion, its position is arbitrary. Three end-support arrangements were considered: fixed-fixed, fixed-pinned, pinned-pinned. The beam has cross-sectional inhomogeneity and the cross-sections are uniform.

For these three-point boundary value problems, the definition of the Green function was discussed. The related Green functions were given in closed-form for the selected problems. The eigenvalue problems for the critical load were replaced by eigenvalue problems governed by homogeneous Fredholm integral equations with kernels derived from these Green functions. Then the latter eigenvalue problems were converted to algebraic form using the boundary element method. In limiting cases regarding the position of the intermediate support and the spring stiffness, the classical results for fixed-fixed, fixed-pinned, pinned-pinned beams were found.

Thesis 2.

I have investigated the free vibrations of straight uniform beams with four supports. The free vibration problems I attacked were considered as four-point boundary value problems. To find solutions, I generalized the concept of the Green function for four-point boundary value problems, extending the definition that was introduced for three-point boundary value problems. All the properties and construction steps concerning these Green functions have been clarified.

The Green functions were provided in closed-form for straight beams with cross-sectional inhomogeneity that have two intermediate roller supports at arbitrary positions, while the ends are either fixed-fixed or pinned-roller supported. With the related Green functions as kernels, the assigned four-point eigenvalue problems were transformed to homogeneous Fredholm integral equations. These were replaced by algebraic eigenvalue problems, which were then solved numerically. In limiting cases regarding the position of the intermediate two supports, the classical results for fixed-fixed, fixed-pinned, pinned-pinned beams are found.

Thesis 3.

The Green function definition was extended to coupled boundary value problems, when the differential operators have the same order but are different in the two domains. The properties and construction steps were clarified in details. With the introduced method, I investigated the free and loaded vibrations together with the stability of stepped straight beams with cross-sectional heterogeneity. As regards the end supports, the beams are fixed-fixed or pinned-pinned. The location of the step is arbitrary. The concentrated axial load is exerted at one end of the beam.

The Green functions were determined in closed-form for compressive and tensile axial load as well. The stability problem was analyzed under a compressive load, while the loaded vibration frequencies were examined for both compression and tension cases. The ratio of the natural circular frequency square for the axially loaded beam and of the free vibrations (no axial load on the beam) increases for tensile load, and decreases for compressive load. This relationship, as a function of the load magnitude, is almost linear.

Thesis 4.

I have investigated the in-plane limit-point stability of shallow arches. The perfect arches are circular but have initial geometrical imperfections. I incorporated the effect of arbitrary initial displacements to the mechanical model which is geometrically non-linear. The imperfections cause initial strains but the initial stresses were neglected. The static equilibrium equations were found with the principle of virtual work. I developed analytical solutions for some symmetric and anti-symmetric imperfection shapes.

I evaluated the results for the buckling loads and in-plane behavior. The lowest mode shape errors have greatest effects on the lowest buckling load for smaller opening angles. This effect can be positive or negative too, depending on the actual imperfection parameters. The equilibrium path can show two kinds of behavior. When snap-through buckling is not possible, there is one stable equilibrium branch. When there exists a buckling load, the number of equilibrium branches is three: one primary and remote stable, with an unstable between.

4 Potential application of the results

The findings of this study have important real-world implications in a variety of engineering domains where structural elements' vibrational properties and stability are critical. The following are some approaches to make use of the created methodologies and findings. Bridge structures and industrial piping systems can greatly benefit from stability analysis of beams with three supports. The analysis demonstrates that strategically placing intermediate supports can enhance the load-bearing capacity, resulting in more robust and stable systems. The vibration results of beams supported by four supports can be used to study the free vibration behavior of bridge decks in multi-span bridges. In this application, ensuring that the beams are designed to resist vibrations is crucial for improving the safety of the bridge. In aerospace and mechanical engineering, the vibration analysis of heterogeneous and stepped beams contributes to the design of lightweight yet stable components. The research findings on eigenvalue problems and vibration frequencies can help in the development of robotic arms. Examining the in-plane stability of arches with geometric imperfections offers important information for building tunnels and bridges, where performance and safety can be greatly impacted by initial shape errors.

5 The candidate's related publications

Journal articles

(1) L.P. kiss, G. Szeidl and A. Messaoudi: Vibration of an axially loaded heterogeneous pinned-pinned beam with an intermediate roller support. *Journal of Computational and Applied Mechanics*, **16**(2), pp 99-128, (2021). https://doi.10.32973/jcam.2021.007

(2) A. Messaoudi and L. P. Kiss: A short review on the buckling of compressed columns. *Multidiszciplináris Tudományok: A Miskolci Egyetem Közleménye*, **12**(1), pp 103-111, (2022). https://doi.org/10.35925/j.multi.2022.1.9

(3) L. P. Kiss, G. Szeidl and A. Messaoudi: Stability of heterogeneous beams with three supports through Green functions. *Meccanica*, **57**(6), pp 1369-1390, (2022). https://doi.org/10.1007/s11012-022-01490-z

(4) A. Messaoudi and L. P. Kiss: Buckling of beams with a boundary element technique. *Journal of Engineering Studies and Research*, **28**(2), pp 25-32, (2022). https://doi.org/10.29081/jesr.v28i2.003

(5) L. P. Kiss, G. Szeidl and A. Messaoudi: Vibration of an axially loaded heterogeneous fixed–fixed beam with an intermediate roller support. *Journal of the Brazilian Society of Mechanical Sciences and Engineering*, **44**(10), Paper 461, 19 p, (2022). https://doi.org/10.1007/s40430-022-03732-3

(6) L. P. Kiss, A. Messaoudi and G. Szeidl: Stability of heterogeneous beams with three supports—solutions using integral equations. *Journal*

of Applied Mechanics, **4**, pp 254–286, (2023).
https://doi.org/10.3390/applmech4010015

(7) L. P. Kiss, A. Messaoudi and G. Szeidl: Solutions for the vibration and stability problems of heterogeneous beams with three supports using Green functions. *Journal of Computational and Applied Mechanics*, **17**(2), p 125-155, (2022). https://doi.org/10.32973/jcam.2022.007

(8) A. Messaoudi and L. P. Kiss: Vibration and stability of curved beams using the finite element method. *Annals of Faculty of Engineering Hunedoara: International Journal of Engineering*, **21**(2), p 119-124, (2023).

(9) A. Messaoudi, L. P. Kiss and G. Szeidl: Green functions for four-point boundary value problems with applications to heterogeneous beams. *Journal of Applications in Engineering Science*, **17**, paper 100165, 16 p, (2024).https://doi.org/10.1016/j.apples.2023.100165

(10) A. Messaoudi and L. P. Kiss: In-plane and out-of-plane stability of curved beams – an overview. *Acta Technica Corviniensis – Bulletin of Engineering*, **16**(3), pp 31-36, (2023).

(11) A. Messaoudi and L. P. Kiss: Investigations on the limit-point buckling of curved beams. *Multidiszciplináris Tudományok: A Miskolci Egyetem Közleménye*, **13**(2), pp 78-86, (2023).
https://doi.org/10.35925/j.multi.2023.2.7

(12) L. P. Kiss and A. Messaoudi: Assessments of the non-linear instability of arches with imperfect geometry. *Structures*, **71**, paper 108031, 17 p, (2025). https://doi.org/10.1016/j.istruc.2024.108031

(13) A. Messaoudi: Green functions for coupled boundary value problems with applications to stepped beams made of heterogeneous material. *Journal of Computational and Applied Mechanics*, **19**(2), pp 105–137, (2024). https://doi.org/10.32973/jcam.2024.005

Conference papers

(14) A. Messaoudi and L. P. Kiss: Numerical buckling analysis of straight beams. *8th International Mardin Artuklu Scientific Researches Conference*, in Bhuvaneswari, Gulnaz Gafurova (eds.): *The Book of Full Texts on Applied Sciences*, Mardin, Turkey, **1**, 429 p, pp 1090-1095, (2022).

(15) A. Messaoudi and L. P. Kiss: Buckling of beams by means of a Green function technique. *II. Başkent International Conference on Multidisciplinary Studies*, Ankara, Turkey, pp 197-203, (2022)

(16) A. Messaoudi: Green functions for some beam problems. *Diáktudomány: A Miskolci Egyetem Tudományos Diákköri Munkáiból*, **14**, pp 78-85, (2022).

(17) A. Messaoudi and G. Szeidl: Stability investigation using Green's functions. *Doktoranduszok Fóruma*, pp. 1-7, (2022).

(18) G. Szeidl, L. P. Kiss and A. Messaoudi: Vibration of axially loaded heterogeneous beams with three supports. *The 9th International Conference on "Advanced Composite Materials Engineering" – COMAT 2022*, Brașov, Romania, Transilvania University of Brașov, p 147, (2022).

(19) A. Messaoudi and G. Szeidl: A novel approach to the vibration problem of some stepped beams. *Doktoranduszok Fóruma 2022*, Miskolc-Egyetemváros, Hungary: University of Miskolc, pp 1-6, (2023).

References

[1] D. Gönczi. Finite element investigation in the forming process of aluminium aerosol cans. *Acta Technica Corviniensis - Bulletin of Engineering*, **13**(4):19–22, 2020.

[2] A. Baksa, D. Gönczi, L. P. Kiss, P. Z. Kovács, and Z. Lukács. Experimental and numerical investigations on the stability of cylindrical shells. *Journal of Engineering Studies and Research*, **26**(4):34–39, 2020.

[3] C. M. Wang, C. Y. Wang, and J. N. Reddy. *Exact solutions for buckling of structural members*. CRC Press, 2004.

[4] K. Murawski. *Technical Stability of Continuously Loaded Thin-walled Slender Columns*. Lulu Press, 2017.

[5] A. M. Wahrhaftig, K. M. M. Magalhães, R. M. L. R. F. Brasil, and K. Murawski. Evaluation of mathematical solutions for the determination of buckling of columns under self-weight. *Journal of Vibration Engineering & Technologies*, **4**(1):233–253, 2020.

[6] G. Green. *An Essey on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism*. Notthingam: Printed for the author by T. Wheelhouse, 1828.

[7] M. Bocher. Boundary problems and Green's functions for linear differential and difference equations. *Annals of Mathematics*, **13**(1/4):71–88, 1911-1912.

[8] E. L. Ince. *Ordinary Differential Equations*. Longmans, Green and Co., London, 1926.

[9] L. Collatz. *Eigenwertaufgaben mit Technischen Anwendungen*. Akademische Verlagsgesellschaft Geest & Portig K.G., 1963. Russian Edition in 1968.

[10] L. Collatz. *The Numerical Treatment of Differential Equations.* Springer-Verlag, Berlin-Heidelberg GMBH, 3rd edition edition, 1966.

[11] G. Szeidl. *Effect of the change in length on the natural frequencies and stability of circular beams.* PhD thesis, Department of Mechanics, University of Miskolc, Hungary, 1975. (in Hungarian).

[12] Z. Zhao. Solutions and Green's functions for some linear second-order three-point boundary value problems. *Computers & Mathematics with Applications*, **56**:104–113, 2008.

[13] I. Y. Karaca. Fourth-order four-point boundary value problem on time scales. *Applied Mathematics Letters*, **21**:1057–1063, 2008.

[14] B. Liu. Positive solutions of a nonlinear four-point boundary value problems in Banach spaces. *Journal of Mathematical Analysis and Applications*, **305**:253–276, 2005.

[15] X. Zhang and L. Liu. Positive solutions of fourth-order four-point boundary value problems with p-Laplacian operator. *Journal of Mathematical Analysis and Applications*, **336**:1414–1423, 2007.

[16] Y. Lin and J. Lin. Numerical method for solving the nonlinear four-point boundary value problems. *Communications in Nonlinear Science and Numerical Simulation*, **15**:3855–3864, 2010.

[17] A. Barari, H. D. Kaliji, M. Ghadimi, and G. Domiarry. Non-linear vibration of Euler-Bernoulli beams. *Latin American Journal of Solids and Structures*, **8**:139–148, 2011.

[18] K. Saurabh. Vibration analysis of non-uniform axially functionally graded beam resting on pasternak foundation. *Materialstoday: Proceedings*, 2022.

[19] G. S. Pavan, H. Muppidi, and J. Dixit. Static, free vibrational and buckling analysis of laminated composite beams using isogeometric collocation method. *European Journal of Mechanics - A/Solids*, **96**:104758, 2022.

[20] M. A. De Rosa, P. M. Bellés, and M. J. Maurizi. Free vibrations of stepped beams with intermediate elastic supports. *Journal of Sound and Vibration*, **181**(5):905–910, 1995.

[21] W. Jong-Shyong and C. Bo-Hau. Free vibration of axial-loaded multi-step timoshenko beam carrying arbitrary concentrated elements using continuous-mass transfer matrix method. *European Journal of Mechanics - A/Solids*, **38**:20–37, 2013.

[22] M. A. Mahmoud. Free vibrations of tapered and stepped, axially functionally graded beams with any number of attached masses. *Engineering Structures*, **267**:114696, 2022.

[23] Y. X. Li and J. Gong. Free and forced vibration analysis of general multiple beam systems. *International Journal of Mechanical Sciences*, **235**:107716, 2022.

[24] A. Bizzi, E. L. Fortaleza, and T. S. N. Guenka. Dynamics of heavy beams: Closed-form vibrations of gravity-loaded rayleigh–timoshenko columns. *Journal of Sound and Vibration*, **510**:116259, 2021.

[25] X. Zhao, W.D. Zhu, and Y.H. Li. Analytical solutions of nonlocal coupled thermoelastic forced vibrations of micro-/nano-beams by means of green's functions. *Journal of Sound and Vibration*, **481**:115407, 2020.

[26] H. Hesheng, C. Dengqing, and L. Lun. Green's functions for forced vibration analysis of bending-torsion coupled timoshenko beam. *Applied Mathematical Modelling*, **45**:621–635, 2017.

[27] A. R. Liu, Z. C. Yang, M. A. Bradford, and Y.–L. Pi. Nonlinear dynamic buckling of fixed shallow arches under an arbitrary step radial point load. *Journal of Engineering Mechanics*, **144**(4):04018012, 2018.

[28] A. R. Liu, Y. H. Huang, J.Y. Fu, Q. C. Yu, and R. Rao. Experimental research on stable ultimate bearing capacity of leaning–type arch rib systems. *Journal of Constructional Steel Research*, **114**:281–292, 2015.

[29] G. J. Simitses. *An Introduction to the Elastic Stability of Structures*. Prentice-Hall, 1976.

[30] S. P. Timoshenko and J. M. Gere. Theory of elastic stability. *McGraw Hill*, 1961.

[31] P. R. Calboun and D. A. DaDeppo. Nonlinear finite element analysis of clamped arches. *Journal of Structural Engineering*, **109**(3):599–612, 1983.

[32] Z. M. Elias and K. L. Chen. Nonlinear shallow curved beam finite element. *Journal of Engineering Mechanics*, **114**(6):1076–1087, 1988.

[33] Y.–L. Pi and M. A. Bradford. In-plane strength and design of fixed steel i-section arches. *Engineering Structures*, **26**(3):291–301, 2004.

[34] M. A. Bradford, T. Wang, Y-L. Pi, and R. I. Gilbert. In-plane stability of parabolic arches with horizontal spring supports. I: Theory. *Journal of Structure and Engineering*, **133**(8):1130–1137, 2007.

[35] Y. B. Yang and M. A. Shieh. Solution method for nonlinear problems with multiple critical points. *AIAA Journal*, **28**(12):2110–2116, 1995.

[36] S. R. Kuo and Y. B. Yang. Tracing postbuckling paths of structures containing multi loops. *International Journal of Numerical Methods in Engineering*, **38**(23):4053–4075, 1995.

[37] L. P. Kiss. Vibrations and stability of heterogeneous curved beams. *PhD Thesis. Institute of Applied Mechanics, University of Miskolc, Hungary*, 2015.

[38] H. L. Schreyer. The effect of initial imperfections on the buckling load of shallow circular arches. *Journal of Applied Mechanics*, **39**(2):445–450, 1972.

[39] K. Der-Wang and C.K. Mak. Finite element analysis of buckling and post-buckling behaviors of arches with geometric imperfections. *Computers Structures*, **3**(1):149–161, 1973.

[40] T. Furtmüller, C. Adam, and X. Shen. Buckling and post-buckling analysis of three-layer shallow arches with geometric imperfections and interlayer slip. *Thin-Walled Structures*, **193**, 2023.

[41] J. Chroscielewski and V. A . Eremeyev. Can we really solve an arch stability problem? *International Journal of Engineering Science*, 2024.

[42] A. Baksa and I. Ecsedi. A note on the pure bending of nonhomogeneous prismatic bars. *International Journal of Mechanical Engineering Education*, **37**(2):1108–129, 2009.

[43] G. Szeidl and L. P. Kiss. *Green Functions for Three Point Boundary Value Problems with Applications to Beams*, chapter 5, pages 121–161. Nova Science Publisher, Inc.

[44] L. Kiss, G. Szeidl, and A. Messaoudi. Stability of heterogeneous beams with three supports—solutions using integral equations. *Applied Mechanics*, **4**:254–286, 2023.

[45] L. P. Kiss, G. Szeidl, and A. Messaoudi. Vibration of an axially loaded heterogeneous fixed–fixed beam with an intermediate roller support. *Journal of the Brazilian Society of Mechanical Sciences and Engineering*, **44**:461, 2022.

[46] L. Kiss, G. Szeidl, and A. Messaoudi. Stability of heterogeneous beams with three supports through green functions. *Meccanica*, **57**:1369–1390, 2022.

[47] A. Messaoudi, L. P. Kiss, and G. Szeidl. Green functions for four-point boundary value problems with applications to heterogeneous beams. *Applications in Engineering Science*, **17**:100165:16p, 2024.

[48] A. Messaoudi. Green functions for coupled boundary value problems with applications to stepped beams made of heterogeneous material. *Journal of Computational and Applied Mechanics*, **19**(2):105–137, 2024.

[49] L. P. Kiss and A. Messaoudi. Assessments of the non-linear instability of arches with imperfect geometry. *Structures*, **71**:108031, 2025.