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Fuzzy Rule Interpolation and Benchmark System

Ph.D. dissertation

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DECLARATION

The author hereby declares that this dissertation has not been submitted, either in the same or in a different form, to this or to any other university for obtaining a PhD degree. The author confirms that the submitted work is his own and the appropriate credit has been given where reference has been addressed to the work of others.

Miskolc, 2020. July 19
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MA`EN AL-ZU`BI

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LIST OF ABBREVIATIONS

(ATP)	:	Aggregate Then Perform
(BFRI)	:	Backward Fuzzy Rule Interpolation
(CNF)	:	Convex and Normal Fuzzy set
(COA)	:	Center of Area
(COG)	:	Centre of Gravity
(CRF)	:	The Conservation Relative Fuzziness
(CRI)	:	Compositional Rule of Inference
(CW)	:	Computing with Words
(DDOS)	:	Distributed Denial of Service
(dL)	:	Distance Lower
(dU)	:	Distance Upper
(FATI)	:	First Aggregate Then Infer
(FEAT-p)	:	Fuzzy Set interpolation Technique bases on Polar cut
(FEFRI)	:	The Fundamental Equation of The Fuzzy Rule Interpolation
(FIS)	:	Fuzzy Inference System
(FITA)	:	First Infer Then Aggregate
(FIVE)	:	The Fuzzy Interpolation in the Vague Environment
(FL)	:	Fuzzy Logic
(FMP)	:	Fuzzy Modus Ponens
(FMT)	:	Fuzzy Modus Tollens
(FPL)	:	Fixed Point Law
(FRI)	:	The Fuzzy Rule Interpolation
(FRIPOC)	:	Fuzzy Rule Interpolation based on Polar Cuts
(FRIT)	:	Fuzzy Rule Interpolation Toolbox
(GM)	:	General Method
(GMP).	:	Generalized Modus Ponens
(GP)	:	Gergonne Point
(GPL)	:	General Public License
(GPX)	:	Gergonne Point
(HLS)	:	Hotels Location Selection
(IDS)	:	Intrusion Detection Systems
(IMUL)	:	The Improved Multidimensional
(IRI)	:	Individual Rule-based Inference
(KH)	:	Kóczy - Hirota
(LBP)	:	Local Binary Pattern
(LCB^*)	:	Left Core of the conclusion
(LESFRI)	:	Least Squares based Fuzzy Rule Interpolation
(LFB^*)	:	Left Flank of the conclusion

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(LFS)	:	Left Fuzziness Side
(LOM)	:	Largest of Maxima
(LTB)	:	Left Boundary
(MACI)	:	Modified α-Cut based Interpolation
(MF)	:	membership function
(MIB)	:	Management Information Base
(MOM)	:	Mean of Maxima
(OBS)	:	Observation
(OCTFRI)	:	OCTAVE Fuzzy Rule Interpolation
(PS1)	:	Left Fuzziness Length
(PS3)	:	Right Fuzziness Length
(PTA)	:	Perform Then Aggregate
(PWL)	:	Piecewise Linearity
(R)	:	Fuzzy Relation
(RCB^*)	:	Right Core of the conclusion
(RF.PS)	:	Ration Fuzziness of the Fuzziness Side
(RFB^*)	:	Right Flank of the conclusion
(RP)	:	Reference Point
(RTB)	:	Right Boundary
(SCM)	:	The Solid Cutting Technique
(SinglMF)	:	Singleton Membership Function
(SNMP)	:	Simple Network Management Protocol
(SOM)	:	Smallest of Maxima
(SURE-p)	:	The Single Rule Reasoning based on polar cuts
(TCs)	:	The Trilinear Coordinates
(TFR)	:	Transformation of the Fuzzy Relation
(trapmf)	:	Trapezoidal Membership Function
(trimf)	:	Triangular Membership Function
(TSK)	:	Takagi-Sugeno-Kang
(VEIN)	:	Vague Environment based Set Interpolation
(ce)	:	Cylindrical Extension

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INTRODUCTION

The fuzzy set theory offers one distinguished approach to describing and dealing with uncertain knowledge and data, which sets the main basis of approximate reasoning [65], [130]. Compared with the other principal techniques in approximate reasoning, the main advantage of fuzzy sets is that they can conserve interoperability and clarity during the reasoning process due to the use of linguistic terms in fuzzy logic [86], [87], [88], [89], [131]. The fuzzy set theory allows for the addition of vague human assessments in computing problems. Also, it presents an effective means for conflict resolution of multiple criteria and better assessment of options. Fuzzy sets have become an increasingly familiar methodology for the modeling of various kinds of common sense reasoning, especially when dealing with nonlinear, uncertain, vague, partially true and complex systems, such as information processing [132], [133], [134], mechanical control [87], [135], [136], classification tasks [137], [138], natural language processing [139], [140], expert systems [141], [142], image recognition [143], [144], diagnosis [145], [146], [147] and intelligent decision support systems [148], [149], [150].

In a fuzzy inference system, the reasoning techniques are implemented by the execution of fuzzy If-Then rules, which is called a fuzzy rule base. If a fuzzy inference system has a large number of rules that used to cover all possible input antecedents, then its rule base is called "dense" fuzzy-rule base [36], [37]. In this situation, the inference process is relatively straightforward, and any classical inference approach, such as the compositional rule of inference (CRI) [65], [71], can be used to infer the results. In contrast, if fuzzy rule bases do not cover all possible input antecedents, then the fuzzy-rule base is called "a sparse fuzzy-rule base". In this situation, the inference process is more complex, and a more intuitive approach such as a Fuzzy Rule Interpolation (FRI) [3], [16], [25], [78], [91], [92] can use to infer the conclusion in case a sparse fuzzy rule base. In both cases, the rule base and its rules rest prime importance and affect the fuzzy inference system’s accuracy.

FRI approaches are considerably useful for reasoning in case a sparse rule base. The FRI inference systems are based on an interpolation concept that can generate a conclusion from existing rules. The FRI methods can divide into two groups. The first group produces the approximated conclusion from the observation directly. The second group produces the approximated conclusion from the observation base on two steps, (the first step, they interpolate a new rule that antecedent part at least overlaps the observation, the estimated conclusion determined in the second step based on the similarity between the observation and the antecedent part of the new rule). Because the FRI methods were developed independently and proposed theoretically, most of the FRI methods have no practical application; it has been developed a toolbox that includes a set of fuzzy interpolation methods under the Matlab environment. Besides that, a collection of conditions suggested to be a baseline to compare and evaluate the performance of existing and upcoming FRI methods; still, most of the FRI methods not fulfilled some of the suggested conditions which could be useful important for the FRI concept. Because there is no common dataset that is suitable for comparing between FRI methods (e.g., benchmarks examples), all these reasons are leading us to the main aim of the research.

1.1. The Aim of Research

This research is divided into three goals:

- To introduce a brief description of the refreshed and extended version of the original FRI Toolbox under the MATLAB environment, to present the extension of the FRI toolbox under the OCTAVE environment, different examples used to prove the validity of both of FRI toolboxes.
- Several conditions and criteria have been suggested for the fuzzy interpolation concept to enable researchers to evaluate and compare FRI methods; therefore, this research aimed to generate the initial benchmark system (benchmark examples) related to the fuzzy set of the conclusion that must preserve a Piecewise Linearity (PWL) and must produce Convex and Normal Fuzzy (CNF). Hence, these benchmark examples could use to be a baseline for testing other FRI methods against situations that are not satisfied with the linearity and normality conditions for Kóczy - Hirota (KH)-FRI method.
- To develop a new method for fuzzy inference, which is based on the Incircle of a triangular fuzzy number. This approach is suitable in case sparse fuzzy rule bases. It can handle the problems in some exists FRI methods and to be satisfied with most of the FRI properties (such as normality, linearity, multi antecedent variables, approximation capability, extrapolation, Etc.).

1.2. Dissertation Guide

In this subsection, we outline the structure of the remainder of the dissertation.

Chapter 2: We provide a background introduction to Fuzzy Logic, Fuzzy Inference Systems (FISs), Compositional Rule of Inference (CRI), Fuzzy Inference Methods, Fuzzy Rule Interpolation (FRI), and properties of the fuzzy rule interpolation concept.

Chapter 3: In the first part, we provide a comprehensive review of typical FRI methods developed in the last two decades. Johanyák et al. [24] developed one trial for setting up a common FRI toolbox framework in 2006 in the MATLAB FRI Toolbox for systematically evaluating the performance of each method provided, and each method is then tested and compared accordingly. In the second part, we focus on the use of the Fuzzy Rule Interpolation (FRI) methods to support GNU/OCTAVE program. The OCTAVE Fuzzy Rule Interpolation (OCTFRI) toolbox is an open-source toolbox for OCTAVE that provides a large subset of the functionality MATLAB compatible. The OCTFRI toolbox includes functions that enable the user to evaluate Fuzzy Inference Systems (FISs) from the command line and OCTAVE scripts, read/write FISs, and OBS to/from files, and produce a graphical output of both the membership functions and the FIS outputs.

Chapter 4: We introduce fuzzy interpolative reasoning method called "Incircle-FRI" for a sparse fuzzy rule-based system based on the Incircle of a triangular fuzzy number. The suggested method defined for triangular CNF fuzzy sets for a single antecedent universe and two surrounding rules from the rule-base. The chapter also extends the suggested "Incircle-FRI" to trapezoidal-shaped fuzzy set by decomposing their shapes into two triangulars. The generated conclusion is also a CNF fuzzy set. The performance of the suggested method is evaluated based on arbitrary examples and a comprehensive comparison to other current FRI methods.

Chapter 5: We present the extensions of the proposed Incircle-FRI method; firstly, applying to a Hexagonal shaped fuzzy set by decomposing its shape into multiple triangulars. Secondly, to apply with multiple fuzzy rules and multiple antecedent fuzzy interpolative reasoning, thirdly, to be suitable in the extrapolation by modification of the weight derivation and the introduction of the shifting process. All the conclusions of the extensions are producing CNF fuzzy sets. The extensions Incircle-FRI method's performance is evaluated based on arbitrary examples and a comprehensive comparison to some current FRI methods.

Chapter 6: Some several properties and criteria have been suggested for unifying the standard requirements of the FRI methods have to satisfy. One of the most common properties is the demand for Convex and Normal (CNF) and Piecewise Linear (PWL) fuzzy conclusion. The KH-FRI method is the one, which cannot fulfill these properties. Therefore, this chapter aims:

To introduce a survey study using different arbitrary examples to compare FRI methods (KH, KH Stabilized, MACI, IMUL, CRF, VKK, GM, FRIPOC, LESFRI, and SCALEMOVE). Where a set of features were used for this comparing: (No. of Dimensions, Type of Membership Functions, and No. of Membership Functions for the antecedent and consequent). These arbitrary examples classified the FRI methods based on the criteria of the "normality" and "linearity" properties, to highlight some basic problematic properties of the KH Fuzzy Rule Interpolation method with Convex and Normal Fuzzy (CNF) and Piecewise Linearity (PWL) properties.

To set up a brief benchmark, which is suitable to be a baseline for testing other FRI methods against cases that the KH-FRI is not satisfied with CNF and PWL properties. All benchmark examples in this chapter constructed using functions implemented by the MATLAB FRI Toolbox, which provides an easy-to-use framework to compare the conclusions of different FRI methods. The CNF benchmark examples used to compare the KHstabilized, MACI, VKK, and CRF methods. The PWL benchmark examples applied to the KHstab, VKK, FRIPOC, and VEIN-FRI methods. In comparison, the results of the proposed Incircle-FRI method also appearing in the benchmarks.

Chapter 7: We summarize the key contributions of the dissertation and the main scientific results of the research.

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2. Fuzzy Systems and Fuzzy Rule Interpolation Background

This chapter provides a short background of fuzzy logic, fuzzy inference systems, fuzzy knowledge base, compositional rule of inference, Mamdani and Sugeno inference methods, and fuzzy rule interpolation. It also presents an overview of the properties and criteria of the fuzzy rule interpolation concept.

2.1. Fuzzy Logic

Fuzzy logic is a mathematical approach to problem-solving; it performs exceptionally in producing exact results from imprecise or incomplete data. A fuzzy set is different from a crisp set in that it allows its elements to have a degree of membership [64]. The essence of a fuzzy set is its membership function, which defines the relationship between a value in the set's domain and its degree of membership. According to the original idea of Zadeh [65], the membership of an element x to a fuzzy set A, denoted as $\mu_A(x)$, can vary from *"0"* (full non-membership) to *"1"* (full membership), i.e., it can assume all values in the interval [0,1]. The value of $\mu_A(x)$ describes a degree of membership of x in A. Clearly, a fuzzy set is a generalization of the concept of a classical set, which the membership function can only take two values *"0"* and *"1"*. **Fig. 1** describes the difference between the fuzzy set and the crisp set.

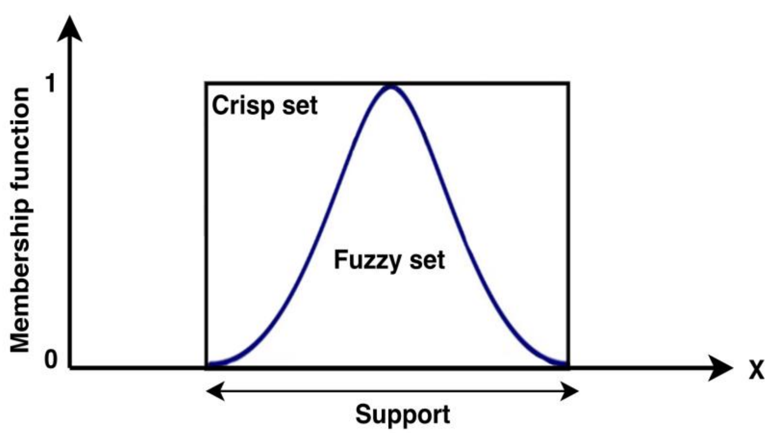

Fig. 1. *The Fuzzy Set and Crisp Set*

In general, a fuzzy system is any system whose variables (or, at least, some of them) range over states that are fuzzy numbers rather than real numbers. These fuzzy numbers may represent linguistic terms such as "*very small*", "*medium*", and so on, as interpreted in a particular context, in this case, the variables are called linguistic variables. The main goal to use the linguistic variables rather than numbers is that linguistic characterizations are, in general, less specific than numerical ones, but much closer to the way that humans formulate and use their knowledge.

An example of a linguistic variable, as shown in **Fig. 2** its name "*Height*", which captures the meaning of the associated base a variable that expresses the Height by real numbers (for example in the interval *[0, 1]*), linguistic values (fuzzy values) of the linguistic variable are {*Very Low, Low, Medium, High, Very High*}. Each of these linguistic terms assigned one of the triangular-shaped fuzzy numbers by a semantic rule, as shown in **Fig. 2**, it is clear that a crucial aspect that will determine the validity of a paradigm of Computing with Words *(CW)*, which refers to a collection of human knowledge expressed in natural language that used to determine correct membership functions for the fuzzy values.

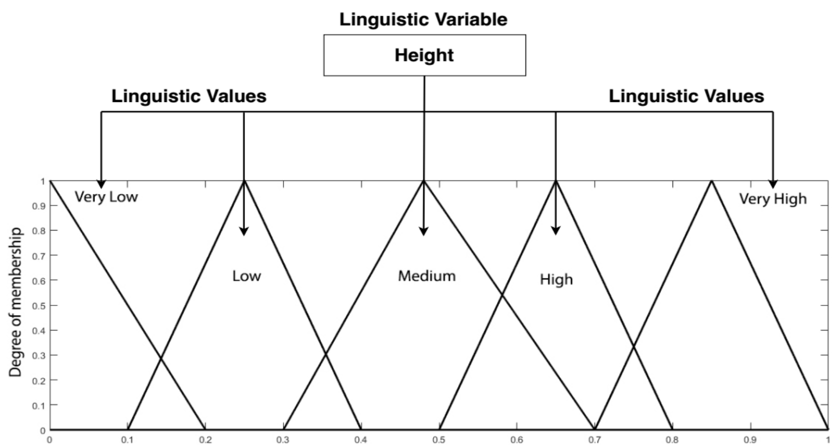

Fig. 2. *The Linguistic Variable "Height"[151]*

2.2. Fuzzy Inference Systems (FIS)

The inference is the process to determine the logical conclusions from assumptions known, considered to be true, or partially true. When the conclusions are determined based on fuzzy linguistic variables using fuzzy set operators (*AND, OR, NOT*), then the process is called approximate reasoning (fuzzy inference). Fuzzy inference is more efficient and useful for those systems, where a system cannot determine in precise mathematical models due to uncertainties, unpredicted dynamics, and other unknown aspects.

Fuzzy Inference Systems (FIS) have been frequently utilized for real-world problems, because of their ability to simulate the human mind's to summarize data and focus on decision-relevant information. The main idea of a typical fuzzy inference system is illustrated in **Fig. 3**, where the standard FIS consists of four interconnected processes [72]. There are two primary components of the fuzzy inference system: a knowledge base and inference mechanism. The FIS crisp input and output, needs two additional blocks; fuzzification and defuzzification. Fuzzification serves to transform a crisp input into a fuzzy value, while defuzzification serves to transform a fuzzy output set into a crisp single value. Therefore, the basic structure of a fuzzy inference system consists of the following components:

- **Fuzzifier**: Which converts the crisp input to a linguistic variable using the membership functions stored in the fuzzy knowledge base.
- **Knowledge Base**: Consists of the set of "Rule Base", "Linguistic Terms (Fuzzy Membership Functions)", and "Inference Parameters", as shown in **Fig. 4**.
- **Inference system**: Which performs the inference procedure upon the rules and given facts to derive an inferred output or conclusion.
- **Defuzzifier**: Which converts the fuzzy output of the inference system to a crisp using membership functions analogous to the ones used by the fuzzifier.

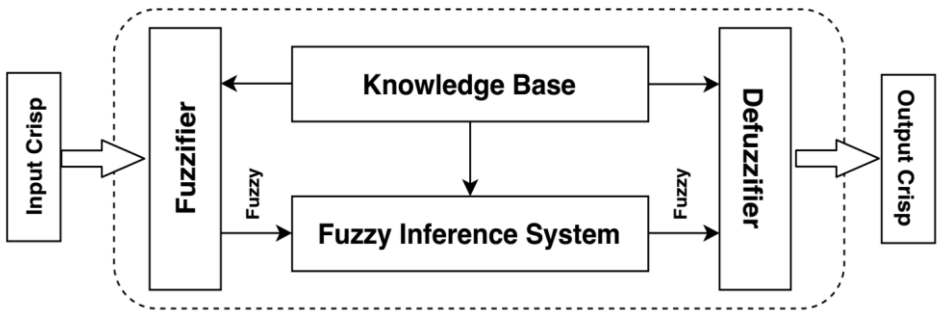

Fig. 3. *The Fuzzy Inference System*

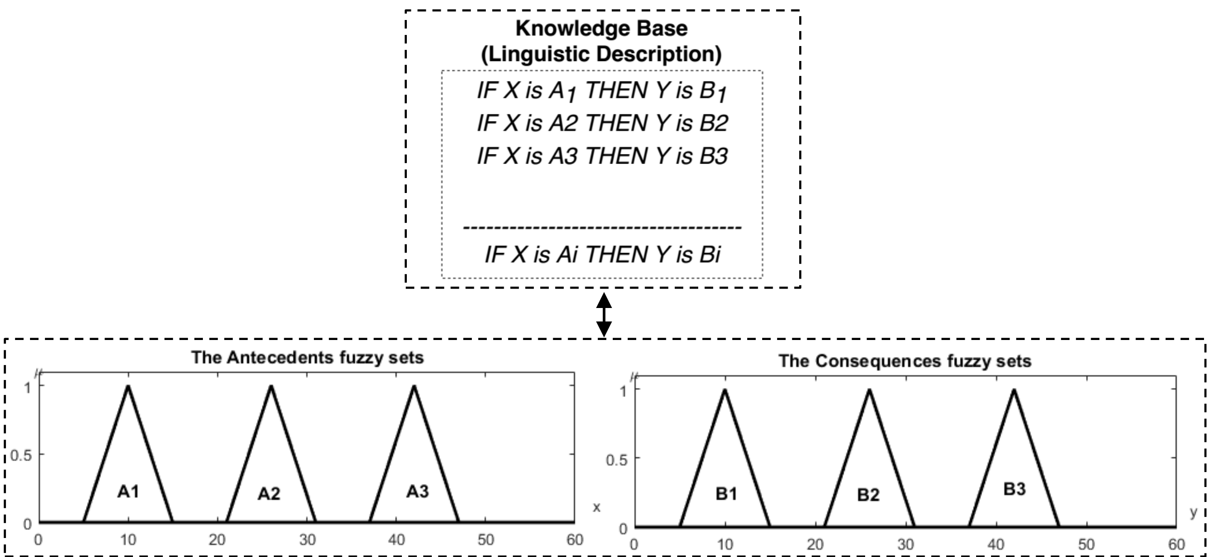

Fig. 4. *The Fuzzy Knowledge Base*

The main part of FIS is the fuzzy rule-based, which is used to represent domain knowledge that is joined with membership functions, in which each rule could be thought of as a subsystem. Rules themselves do nothing unless inputs are applied to them. Fuzzy rules or fuzzy conditional statements are expressions of the form *IF A THEN B*, where *A* and *B* are labels of fuzzy sets [65], which described by appropriate membership functions, also "*If part*" is called the antecedent and "*Then part*" is called the consequent. Fuzzy *IF-THEN* rules are often employed to capture the imprecise forms of reasoning, which play an essential role in the human ability to make decisions in an environment of vagueness.

The antecedent describes to what degree the rule uses, while the consequent assigns a fuzzy function to each of one or more output variables [67], [68], [69]. Fuzzy "If-Then" rules present a simple way to formulate and take human type knowledge because they expressed using linguistic terms. A collection of fuzzy rules base can be obtained from subject matter experts or extracted from data through a rule deducing process.

There are two main methods to construct a rule base for a given problem. The first is by directly translating expert knowledge into rules, and fuzzy inference systems with such rule bases are usually called fuzzy expert systems [1]. Because rules are fuzzy representations of expert knowledge, rule bases offer high semantic interpretability and good generalization capability. However, it is difficult for complex systems to build a rule base, which has led to an alternative approach for rule-based construction. This approach is data-driven, and fuzzy rules are obtained from data using machine learning techniques rather than direct expert knowledge [119], [104], in contrast, rule bases built in this way lack comprehensibility and transparency.

The fuzzy rule base generated in either of the above two ways, several fuzzy inference mechanisms can be used to derive the conclusion from a given observation. The most important model is the compositional rule of inference (CRI), which is introduced in the following subsection.

2.3. Compositional Rule of Inference (CRI)

In approximate reasoning, two inference rules are of major significance: the compositional rule of inference and the generalized modus ponens. The first rule uses a fuzzy relation to represent the connection between two fuzzy propositions explicitly, the second uses an if-then rule that implicitly represents a fuzzy relation. (see example in [66]).

The inference compositional rule (CRI) was first proposed by Zadeh [71] to solve the fuzzy modus ponens (FMP) and fuzzy modus tollens (FMT) models. Later Dubois and Prade [124] introduced two approaches to present the inference of a set of parallel rules for solving the local inference approach, known as IRI, and the global inference approach, known as CRI.

Many of researches have been conducted on CRI and IRI. Such as the "aggregation operator" [125], two methods presented to deal with aggregation operator issues as follows:

- Composition Rule-based Inference (CRI) (the First Aggregate Then Infer (FATI)), where first a combination of all rules from the knowledge base is constructed, and then inference using the supremum-star composition is conducted.
- Individual Rule-based Inference (IRI) (first infer then aggregate: FITA), in which the first step involves inference using the supremum-star composition for each of the rules individually and then, a combination of inference results is performed.

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Researches tried to find an operator for *CRI* and *IRI* aggregation. Assilian [126] used the max-min method. A *"min"* operator is selected as a conjunction in the rule premise and the implication function while a *"max"* operator is used for aggregation. Dubois [127] used a *"min"* operator for aggregation in decision-making. To reduce the computational time, the authors [128], [129] replaced a *MISO* fuzzy rule with an equivalent collection of *SISO* rules with two kinds of aggregation operators.

According to differences between the compositional rule of inference and the generalized modus ponens. Zadeh presented with the treatment of the if-then rule, called the fuzzy conditional statement:

If X is A Then Y is B, and the if ... then ... else rule, if X is A Then Y is B Else Z is C.

Zadeh introduces the idea by two statements as follows:

$$\begin{array}{c} \textit{Observation: } x \textit{ is } A^* \\ \underline{\textit{Rule : If } x \textit{ is } A \textit{ Then } y \textit{ is } B} \\ \textit{Consequence : } y \textit{ is } B^* \end{array}$$

The meaning of the second statement should be defined as a fuzzy relation *R*. With this relation, Zadeh introduces the compositional rule of inference:

if *R* is a fuzzy relation from *X* to *Y*, and A^* is a fuzzy subset of *X*, then the fuzzy subset B^* of *Y*, induced by A^* is given by the composition of *R* and A^*, that means,

$$B^*: A^* \; o \; R = A^* \; o \; (A \rightarrow B) \tag{2.1}$$

Where *o* is the composition operator (see **Fig. 5**), in this case, one should take the cylindrical extension (denoted by *(ce)*) of A^*, take the intersection with *R*, and project the result onto *Y-axis*. When *R* is built up from A^* and B^* with the rule

$$R = ce(A^*) \cup B \text{ , i.e., } \mu_R(x,y) = max(1 - \mu_A(x), \mu_B(y)) \tag{2.2}$$

The compositional rule of inference always requires an explicit relation, for example in

Observation: *x is tall*
Relation: *y is a bit shorter than x,*
Conclusion: *y is more or less tall;*

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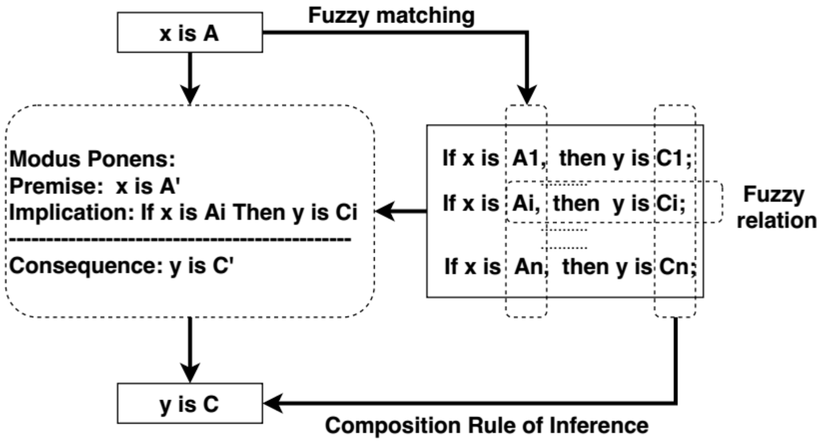

Fig. 5. *Compositional Rule of Inference Example*

In [70], Zadeh proposes the following way to handle the compositional rule of inference. When the inference scheme is:

$$\frac{\begin{array}{c} X_1 \text{ is } A, \\ X_1 \, R \, X_2, \end{array}}{\therefore X_2 \text{ is } B;}$$

where A and B are fuzzy sets representing the meaning of A and B and R is a fuzzy relation defining the meaning of R, and A is defined on X, and R on $X \times Y$, then:

$$B = Proj\ (A \circ R)\ on\ Y, \quad \text{i.e., } \mu_B(Y) = \max_x min(\mu_A(x), \mu_R(x,y)) \qquad (2.3)$$

For more details about this relation, see examples in [66] (pages 84 and 85).

Fuzzy inference is the process of formulating the mapping from a given input to an output using fuzzy logic. The mapping then provides a basis from which decisions can be made or patterns discerned. The process of fuzzy inference involves all of the pieces described so far, i.e., membership functions, fuzzy logic operators, and if-then rules. A number of existing fuzzy reasoning methods based on CRI have been proposed in the literature [36], [120]. In particular, the first successful practical approach was Mamdani inference [1], [121], which is also commonly implemented fuzzy methodology in physical control systems at present [122], [123]. It was originally proposed as an attempt to control a steam engine and boiler combination by synthesizing a set of linguistic control rules obtained from experienced human operators. Mamdani inference implements CRI using minimum as the t-norm operator due to its simplicity. Another approach is called Takagi–Sugeno–Kang (TSK) inference [2], [73], which is introduced in the next subsection.

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2.4. Mamdani and Takagi-Sugeno-Kang Fuzzy Inference Methods

The main difference between the traditional Mamdani inference method and the TSK inference method is the way of the output produced from the fuzzy inputs [100]. While the Mamdani system uses the technique of defuzzification of a fuzzy output, the TSK system uses a weighted average to compute the crisp output. The power expressive and interpretability of Mamdani output reduced in the TSK systems since the consequents of the rules are not fuzzy [74], [76], [93]. However, TSK has better processing time since the weighted average replaces the time-consuming defuzzification process. Due to the interpretable and intuitive nature of the rule base, Mamdani inference systems widely used in particular for decision support applications [75].

The example in [1] explains the working with a model of the fuzzy system, where a simple two inputs x and y (antecedents) and a single output z (consequent) is described by a linguistic *If-Then* rule to describe Mamdani inference method and TSK inference method forms as:

Mamdani inference method:

Rule1: IF X is A_1 OR Y is B_1 THEN Z is C_1

Rule2: IF X is A_2 AND Y is B_2 THEN Z is C_2

TSK inference method:

Rule1: IF X is A_1 OR Y is B_1 THEN Z = f (x, y)

Rule2: IF X is A_2 AND Y is B_2 THEN Z = f (x, y)

Where *X* and *Y* are the antecedent variables, *Z* is consequent variable, respectively, *A* and *B* are fuzzy sets in the antecedent part, *C* is a fuzzy set in the consequent part, *f (x, y)* is a crisp function in the consequent part.

Some of the most popular Mamdani defuzzification techniques are usually a variation of the max criterion method. These include the Smallest of Maxima (SOM), Largest of Maxima (LOM), and the Mean of Maxima (MOM). These methods select the smallest, largest, and mean output value for inputs whose membership value reaches maximum. MOM is one of the most popular methods; it calculates the final output "Z" by averaging the set of output values that have the highest possible degree "M" using the formula given in **Eq.(2.4)** [101].

$$Z = \sum_{i=1}^{l} \frac{x_i}{l}, \qquad x_i \in M. \tag{2.4}$$

Two other commonly used defuzzification techniques are the Center of Gravity (COG) / centroid and Center of Area (COA) / bisector method.

The COG / centroid method determines the crisp output by calculating the center of gravity of the possibility distribution of the output. For continuous values, the output "Z" calculated using **Eq.(2.5)** [101].

$$Z = \frac{\int_z \mu_A(z).zdz}{\int_z \mu_A(z)dz} \qquad (2.5)$$

where Z is the output variable, and $\mu_A(z)$ is the membership function of the aggregated fuzzy set A with respect to z.

The COA is similar to the COG method. However, it calculates the position under the curve where the areas of both sides are equal. The COA can calculate using **Eq.(2.6)** [101].

$$\int^{z} \mu(x)dx = \int_{z} \mu(x)dx \qquad (2.6)$$

Authors in [102] presented a detailed analysis of various defuzzification techniques, including COG and MOM. They concluded that COG yields better results. For this reason, the COG/centroid defuzzification technique used in this work.

The output membership function for the Mamdani scheduler made of triangular membership functions, shown in **Fig. 6**. It consists of 5 fuzzy values, namely: *Very Low (VL), Low (L), Medium (M), High (H), and Very High (VH)*.

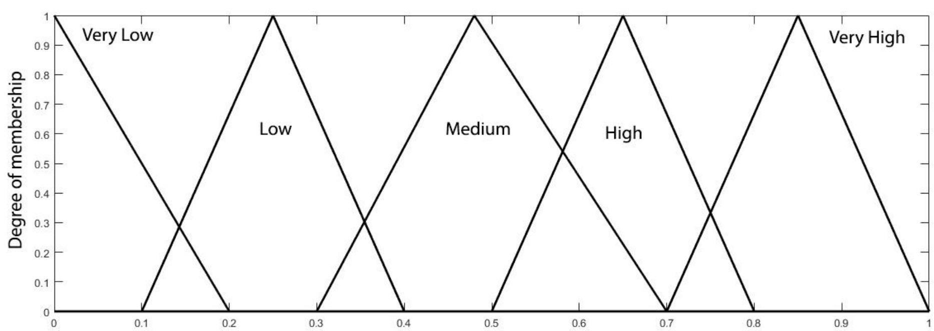

Fig. 6. *Mamdani Consequent Values*

Sugeno FIS uses the weighted average to compute the crisp output, and thus the complex iteration process used by Mamdani is bypassed. The Sugeno FIS does not have an output membership function. The output for Sugeno FIS is shown in **Fig. 7**, and it is a constant value. It consists of five output points, which are the same as the number of membership functions for the Mamdani output (*Very Low (VL), Low (L), Medium (M), High (H), and Very High (VH)*). The Sugeno FIS is a less computationally complex algorithm than the Mamdani equivalent. The interpretability and the expressive power of the Mamdani FIS are lost in the Sugeno FIS because the rules' consequence is not fuzzy [101]. It means that the output will be a constant rather than a fuzzy set when the rules are evaluated. Thus, the impact of this on the system performance will be evaluated.

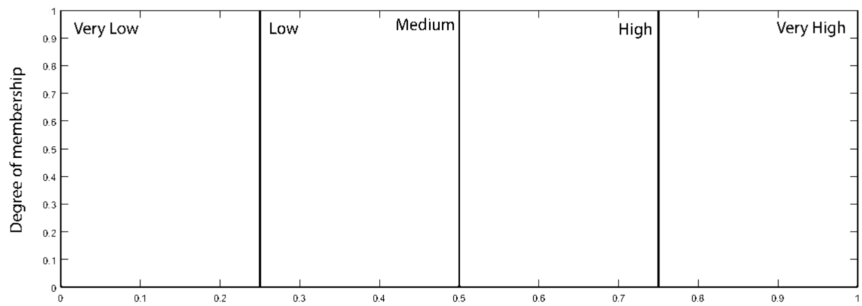

Fig. 7. *Sugeno Consequent Values*

2.5. Fuzzy Rule Interpolation (FRI)

In most fuzzy inference systems, the completeness of the fuzzy rule base is required to generate meaningful output when classical fuzzy inference methods are applied; this emphasizes the need for a complete rule base for the fuzzy inference, which covers all possible inputs. Regardless of how a rule base constructed, be it by human experts or by an automated agent, often incomplete or sparse rule bases are generated.

A complete rule base is especially impracticable in a multidimensional environment, where the number of rules increases exponentially [38], which is based on the input variables and the fuzzy linguistic labels associated with each variable increase. In this situation, the classical fuzzy reasoning techniques cannot generate an acceptable output for such cases; the following example can explain this: Suppose in the assumed sparse fuzzy rule-bases there are only two rules, which given below:

An example related to the ***density*** fuzzy rule bases:

If X is A_1 then Y is B_1
If X is A_2 then Y is B_2
If X is A_3 then Y is B_3
Observation: X is A_1

Conclusion: Y = (B_1)

In another way, using **Eq.(2.7)**:

$$\mathrm{supp}\left(\bigcup_{k=1}^{n} \mathrm{A}_{ik}\right) = \mathbf{X}_i \tag{2.7}$$

An example related to the ***sparsity*** fuzzy rule bases:

If X is A_1 then Y is B_1
If X is A_3 then Y is B_3
Observation: X is A_2

Conclusion: Y = (??)

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In another way, using **Eq.(2.8)**:

$$\operatorname{supp}\left(\bigcup_{k=1}^{n} A_{ik}\right) \neq X_i \tag{2.8}$$

Where X_i is the i^{th} input universe, A_{ik} is the k^{th} set of the partition of X_i and *supp* is the support.

A straightforward solution to handle incomplete or sparse fuzzy rule bases and to infer reasonable output is by the application of FRI methods. FRI techniques initially presented to generate a conclusion in case sparse fuzzy rule bases, which encouraged to extend the usage of fuzzy inference mechanisms for sparse fuzzy rule-based systems [24].

Interpolation is a mathematical term for finding new data points within the range of a discrete set of known data points. Fuzzy rule interpolation (FRI) performs interpolative approximate reasoning by the existing closest fuzzy rules, where there is no matching of fuzzy rules. Generally, these FRI methods are capable of performing two types of inference operation: fuzzy interpolation and fuzzy extrapolation depending on the location of selected closest rules, as shown in **Fig. 8**. If the given input observation lies among the selected closest rules, then fuzzy interpolation operation is performed; otherwise, if the given input observation lies to one side of all selected closest rules, then extrapolation is performed. A comprehensive overview of FRI techniques will be presented later in chapter 3.

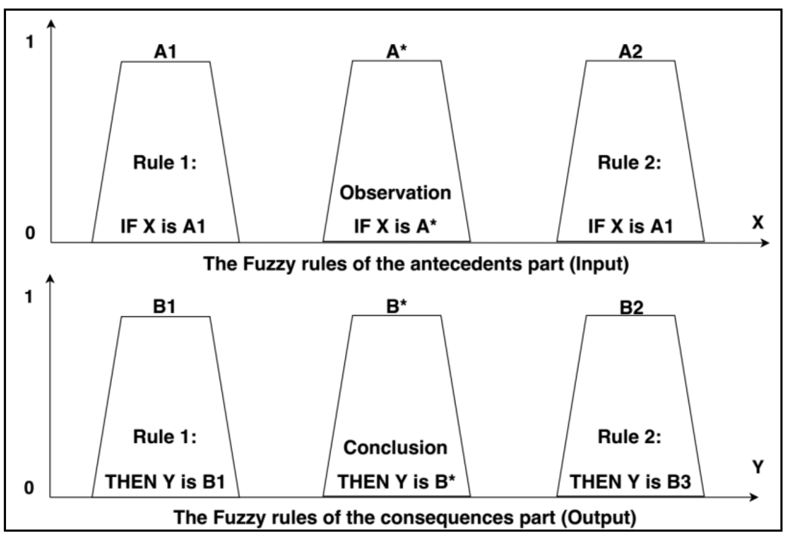

Fig. 8. *The General Illustration of Fuzzy Interpolation [51]*

There are several fuzzy rule interpolation methods (e.g., [3]-[6], [14], [22], [25], [39], [42], [48], [49],[50],[51], [52].) available in the literature. Most of the methods infer directly the conclusions based on observation, such as linear interpolation method [3], [16], [25], [92], Extension KH central point based interpolation method [4], Conservation of the Relative

Fuzziness interpolation method (CRF) [7], Improved Multidimensional modified α-Cut interpolation method (IMUL) [8], improving the possible abnormal conclusion in KH linear interpolation [5], the modified α-Cut based interpolation method [6], and an interpolative reasoning method based on the slopes of fuzzy triangular sets [42].

In contrast, some of the FRI need two steps to calculate conclusions. These methods construct an approximated fuzzy rule based on certain similarity principles and then give the conclusion by approximate transformation, such as Generalized Methodology based interpolation method (GM) [9], Polar Cuts based interpolation method (POC) [10], LEast Squares method based interpolation method (LES) [11] Vague Environment based two-step interpolation method (VE) [63]. Scale and Move transformation based interpolation method (ScaleMove) [22], [85]. The Fuzzy Interpolation in the Vague Environment (FIVE) method [60], [61], [62]. The areas of fuzzy sets interpolation method [52].

2.6. Properties of the Fuzzy Rule Interpolation Concept

While Fuzzy Rule Interpolation (FRI) offers a flexible solution for the problem of sparse fuzzy rule-based inference, there are many aims require careful consideration in devising such systems, which would make the evaluation and comparison of the different techniques based on the same fundamentals possible.

However, according to the existing literature (e.g., [5], [6], [9], [43], [77]) different criteria and properties were defined, which put together considering different points of view according to apply the FRI concept. In particular, it ensures that these methods should: produce the normal conclusion, maintain the piecewise linearity, to apply to different kinds of fuzzy sets, to be able to handle multidimensional environments and to minimize computational complexity [9], [14], [17], [41], [79], [80]-[83].

Based on reviewing a wide range of fuzzy interpolation methods, a set of relevant performance evaluation criteria identified and generalized. However, not all such criteria need to fulfill in developing and applying the FRI methods mentioned. However, it expected that most of the criteria should be satisfied with a useful fuzzy rule interpolation technique with other problem-specific parameters. Therefore, as a step towards the unification, several properties presented to be a baseline to compare and evaluate FRI methods. In the following, we introduce essential FRI properties:

2.6.1. Avoidance of abnormal conclusion [6], [9], [43]

A fuzzy rule interpolation method should produce valid conclusion fuzzy sets; this means that the resultant membership value must be a function of the consequent domain in the range of [0, 1], i.e., the membership function of the conclusion should not be deformed, only one membership function value could address to one element of the conclusion. In the case of the α–cut based FRI methods, this condition can be described according to [6] by the following constraints.

Let X_j $(j = 1,...,n)$ be input dimensions and Y output space, denoting the Cartesian product of input dimensions by $X = X_1 \times X_2 \times \ldots \times X_n$. A fuzzy IF–THEN rule is given as R_i: *if* $A_{i1} \wedge A_{i2} \wedge \cdots \wedge A_{in}$ *then* B_i. Where antecedents $A_{ij} \in F(X_j)$, consequents $B_i \in F(Y)$, and $F(Z)$ denote the entirety of all fuzzy subsets of Z. We denote the (n-dimensional) Cartesian product of antecedents Aij, $(j = 1,...,n)$ of rule R_i by $A_{(i)}$. Therefore, fuzzy set $A \in F(Z)$ is valid if its membership function is valid. α-cuts characterize the validity of the fuzzy set as follows:

$$\forall_\alpha,\ \alpha_1 < \alpha_2 \in (0,1]: inf\{A_\alpha\} \leq sup\{A_\alpha\}\ and \\ inf\{A_{\alpha 1}\} \leq inf\{A_{\alpha 2}\}\ and \\ sup\{A_{\alpha 2}\} \leq sup\{A_{\alpha 1}\} \tag{2.9}$$

where "*inf*" and "*sup*" are the lower and upper endpoints of the actual α-cut of the fuzzy set.

Mapping validity: For each $A^* \in F(X)$ with a valid fuzzy membership function, the conclusion generated by mapping I, $B^* = I(A^*) \in F(Y)$ should also be a valid fuzzy set.

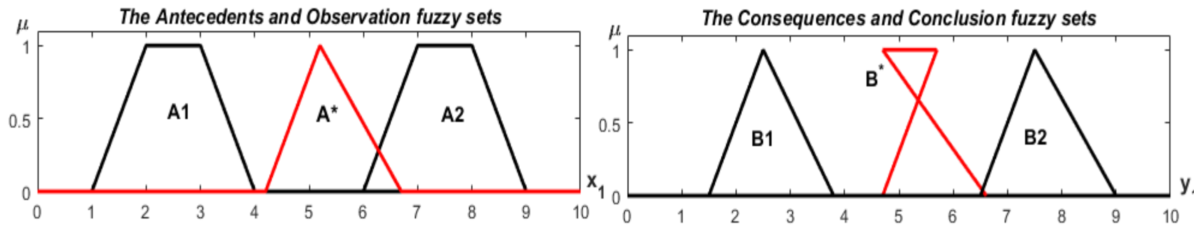

Fig. 9. *The Example Describes Property (2.6.1) with Invalid Conclusion*

For more details, related to this point see (Chapter 6 - Subsection 6.2 demonstrated all notations of CNF condition with examples)

2.6.2. The continuity of the mapping between the antecedent and consequent fuzzy sets [9], [43]

This property interpreted as "*the more similar observation to an antecedent, the more similar conclusion should be to the corresponding consequent of the given antecedent*" in [78].

Let $S_Z: F(Z) \times F(Z) \rightarrow R$ *denote the similarity function defined in the fuzzy sets of Z. Then, for* $A^*, A_1, A_2 \in F(X)$, *if* $S_x(A^*, A_{i1}) \geq S_x(A^*, A_{i2})$ *then* $S_y(I(A^*), B_{i1}) \geq S_y(A^*, B_{i2})$, *where* $R_{ij}: A_{ij} \rightarrow B_{ij}$ $(j=1,2)$ *are two rules of rule base R.*

Many researchers only consider the extreme case of this condition when the observation coincides with a rule antecedent referred to as compatibility with the rule base [9], [43], [82] (*see condition 2.6.3*). In logic, this property is called Modus Ponens (MP). Note that it is also called continuity of the model characterized by the fuzzy relation of the rule base [103].

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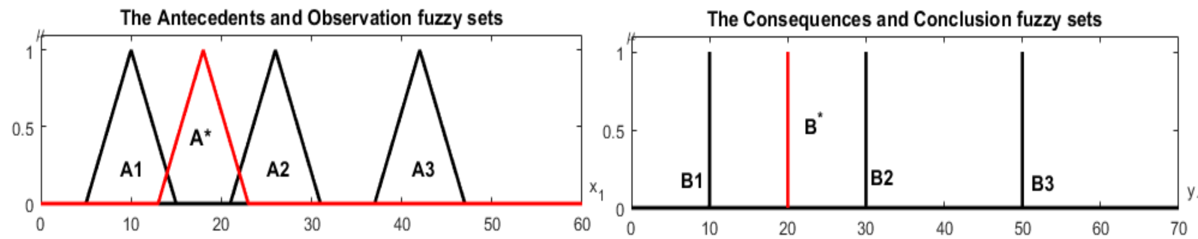

Fig. 10. *The Example Describes Property (2.6.2) with Mapping Between the Antecedent and Consequent*

2.6.3.Compatibility condition of the observation with rule-bases

This condition follows compatibility with the rule base and based on the modus ponens in logic; this means the condition on the validity of the modus ponens, namely if an observation coincides with the antecedent part of a rule, the conclusion produced by the method should correspond to the consequent part of that rule.

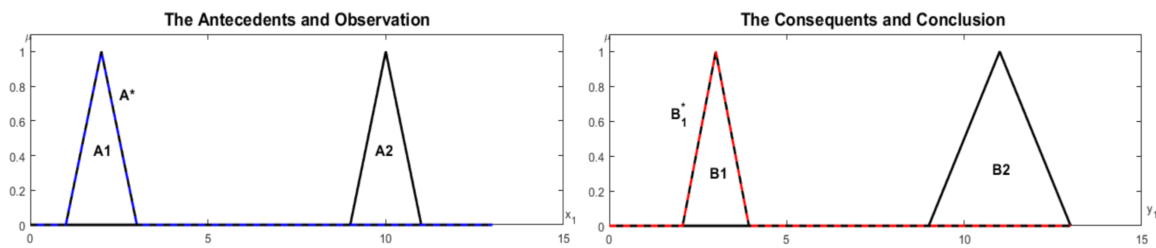

Fig. 11. *The Example Describes Property (2.6.3) with Compatibility with Rule-Bases*

2.6.4. Preservation condition of the observation between adjacent two rule-based [43]

A fuzzy rule interpolation method should keep the neighboring quality of the interpolated result; this implies that if the observation surrounded by the antecedent sets of two adjacent rules, then inferred the consequent sets of those rules must surround conclusion. If the antecedents of the two given rules are A_1 and A_2 and their consequents are B_1 and B_2, the observed rule antecedent A^* should lie between A_1 and A_2 such that the inferred conclusion by interpolation method should fall between the two rules consequents B_1 and B_2.

In linear interpolation, if ($A_{1j} \prec A^*_j \prec A_{2j}$) for all $j=1, ..., n$, then ($B_1 \prec I(A^*) \prec B_2$), where ($R_j$: $A_j \rightarrow B_j$) ($j = 1,2$) are two riles of rule base R and $\prec$ is a partial order operator.

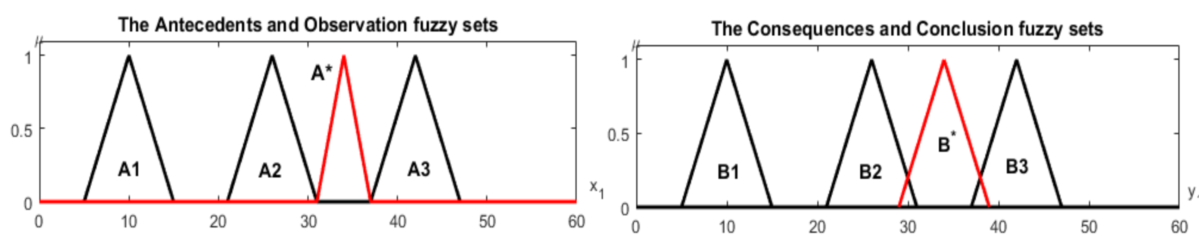

Fig. 12. *The Example Describes Property (2.6.4) with Preservation In-Between*

2.6.5. The fuzziness of the approximated result.

There are two opposite approaches in the literature related to this topic [77].

- Observation (A^*) is a singleton, then conclusion $I(A^*)$ should be a singleton and,

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- All B_I, where (I) denotes the indices of rules that contribute to the calculation of conclusion $I(A^*)$, and observation A^* are singleton, then $I(A^*)$ should be a singleton.

Thus, the crisp conclusion can be expected if all the consequents of the rules taken into consideration during the interpolation are singleton shaped, i.e., the knowledge base produces certain information from fuzzy input data.

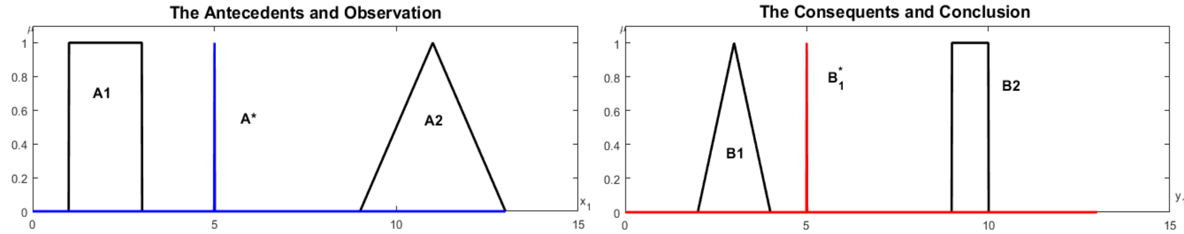

Fig. 13. *The Example Describes Property (2.6.5) with Fuzziness of the Approximated Result for the First Approach*

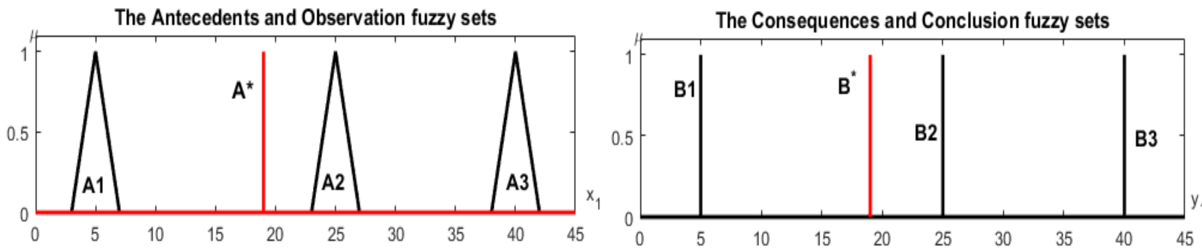

Fig. 14. *The Example Describes Property (2.6.5) with Fuzziness of the Approximated Result for the second Approach*

2.6.6. Approximation capability. [17]

The estimated rule should approximate the possibly highest degree the relationship between the antecedent and consequent universes. If the number of the measurement (knot) points tends to infinite, the result should converge to the approximated function independently from the knot points' position. This condition means the stability between the observation shape and conclusion shape, which must be identical.

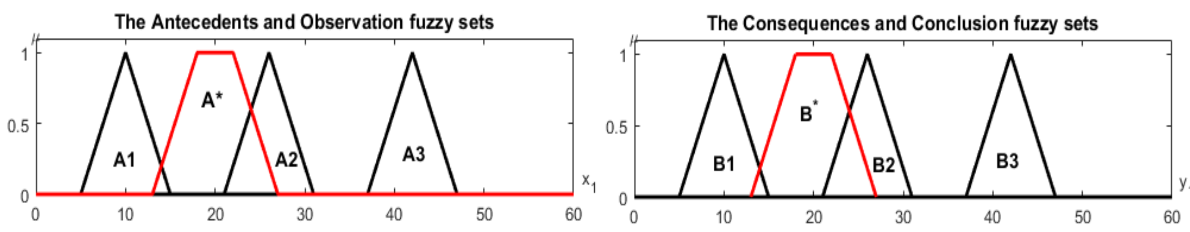

Fig. 15. *The Example Describes Property (2.6.6) with the Approximation Capability*

2.6.7. Preservation of Piecewise Linearity (PWL).

If the fuzzy sets of the rules taken into consideration are piecewise linear, the approximated sets should conserve this feature. This means that a piecewise linear conclusion should be inferred from piecewise linear rules and observations [9], [14], [17], [41], [79], [80], [81], [82],

[83]. Strictly speaking, there must not be any further interpolation other than computing with the odd points only [84], [85]. For more details about this condition (Chapter 6 - Subsection 6.3 demonstrated all notations of PWL condition with examples)

2.6.8. The investigating condition for multidimensional of rule-bases.

Applicability in multidimensional input space: Mapping *I* must be applicable to arbitrary finite dimensions of input space.

Let $R = \{R_i | i = 1,\ldots,r\}$ be a rule base with multidimensional rules of form ***Eq.(2.10)***. *We call mapping $I: F(X) \rightarrow F(Y)$ rule interpolation, if it assigns to each observation $A^* \in F(X)$ an (interpolating) conclusion $I(A^*) = B^* \in F(Y)$.*

$$R_i: \text{If } (A_{i1} \wedge A_{i2} \wedge \cdots \wedge A_{in}) \text{ Then } (B_i) \tag{2.10}$$

Where antecedents $A_{ij} \in F(X_j)$, consequents $B_i \in F(Y)$, and $F(Z)$ denote the entirety of all fuzzy subsets of Z. We denote the (n-dimensional) Cartesian product of antecedents A_{ij}, $(j = 1,\ldots,n)$ of rule R_i by $A_{(i)}$.

In other words, mapping *(I)* must apply to arbitrary finite dimensions of input space. FRI toolbox was initially motivated in attempts to reduce complexity, which is meaningful only in the case of many input dimensions, so the FRI toolbox working only with a one-dimensional rule base has limited applicability. Therefore, a fuzzy rule interpolation method should be able to deal with different kinds of membership functions with different rules. Simply it means that the method should work when the antecedents' fuzzy sets and the consequents of the different fuzzy rules have different kinds of membership functions.

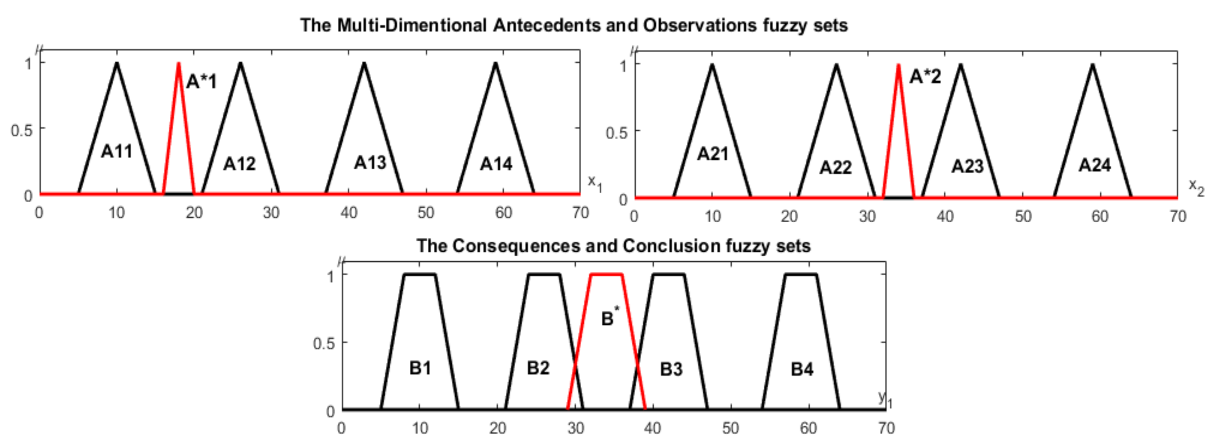

Fig. 16. *The Example Describes Property (2.6.8) with the Multidimensional of Rule-Bases*

2.6.9. Applicability without any constraint regarding the shape of the fuzzy sets.

This condition can be lightened practically to polygons since piecewise linear sets are most frequently encountered in the applications.

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Mapping I must apply to an arbitrary rule base and observation, without constraints regarding fuzzy set shape.

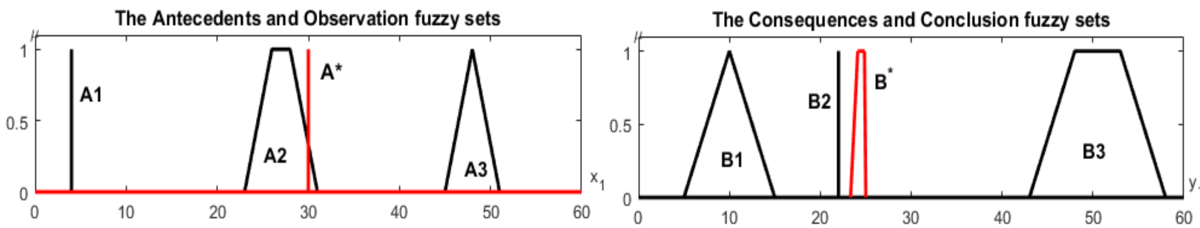

Fig. 17. *The Example Describes Property (2.6.9) with the Applicability of Rule-Bases*

2.6.10. Extrapolation capability of the method.

A method with mapping *I* applies to extrapolation if it generates a conclusion when the observation located in an extrapolative position, which means: If observation A^* located so that A_{i1} and A_{i2} exist such that $(A_{i1} \prec A^* \prec A_{i2})$, then FRI is applied based on rules R_{i1} and R_{i2} to obtain the conclusion. Otherwise, when all rule antecedents A_i $(i = 1, \ldots, r)$ either precede or are preceded by A^*:

$$\forall_i < \in [1,r]: If\ A^* \prec A_1\ or\ A_1 \prec A^* \tag{2.11}$$

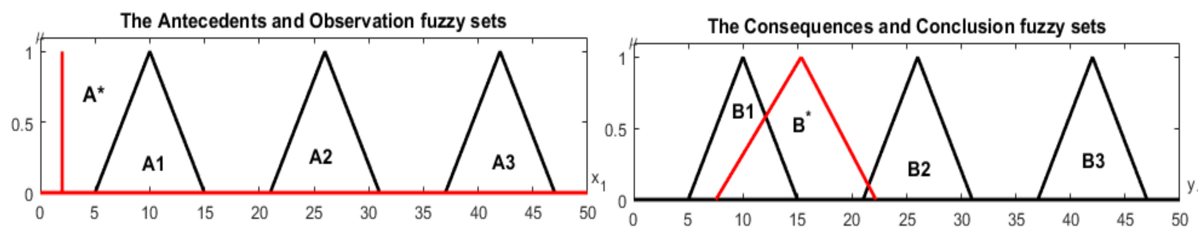

Fig. 18. *The Example Describes Property (2.6.10) with the Extrapolation Property*

2.6.11. Applicability condition in case overlapping antecedent rule-bases.

A fuzzy rule interpolation method should be able to support rules, where antecedents overlap with each other, this means that the method is operable on such a problem, in which two adjacent fuzzy rules have some common members or their intersection are empty.

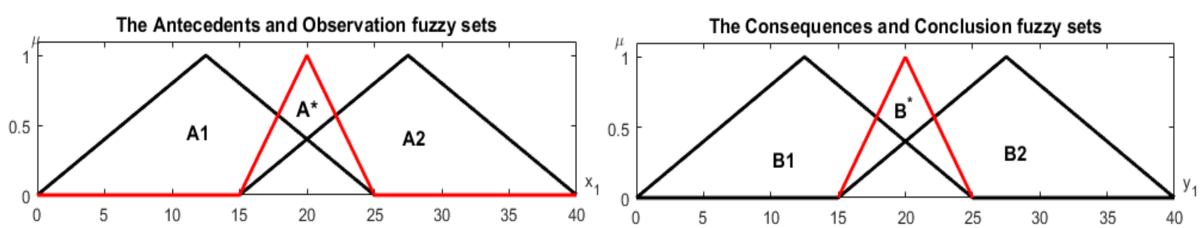

Fig. 19. *The Example Describes Property (2.6.11) with the Overlapping Fuzzy Sets*

2.6.12. Preservation condition of Convexity and Normality.

A fuzzy rule interpolation method should maintain the normality and convexity for any interpolative results; this means that if an observation is normal and convex and all the fuzzy values in the rule-base are also normal and convex, then the interpolated conclusion should also be normal and convex. The normal condition is given below, which shows that at least one element membership function value must be equal to *1:* $(\mu_A(x) = 1, \exists x \in X)$. The convexity condition is given below which dictates that membership function values must be increased or decreased monotonically on either side of the maximum point:

$$\mu_A(\lambda x_1 + (1 - \lambda)x_2) \geq min(\mu_A(x_1); \mu_A(x_2)) \tag{2.12}$$

where $\lambda \in [0, 1], x_1, x_2 \in X$.

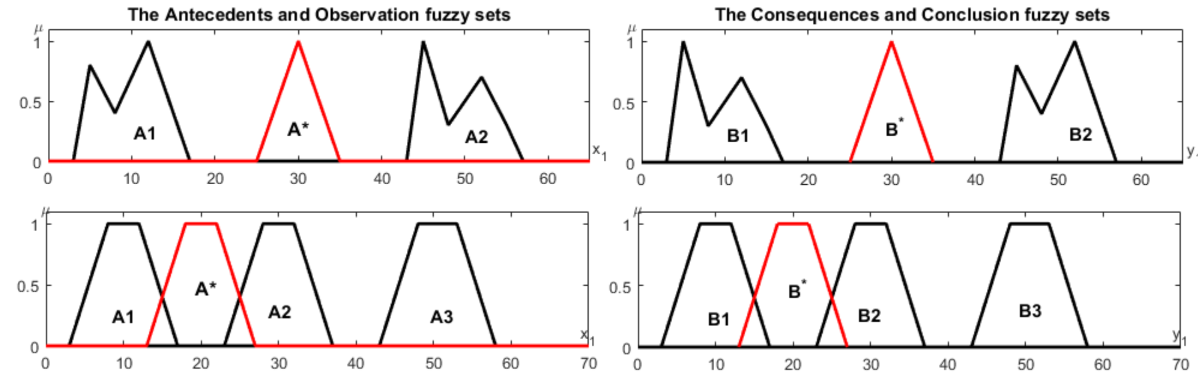

Fig. 20. *The Example Describes Property (2.6.12) with the Convexity Fuzzy Sets*

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INTRODUCTION

In this chapter, we provide a brief overview of the basic definitions of the complete and sparse fuzzy rule-base. We present an overview of the implemented Fuzzy Rule Interpolation (FRI) methods (KH, KHStabilized, MACI, IMUL, CRF, VKK, GM, FRIPOC, LESFRI, and SCALEMOVE). We introduce a brief description of the refreshed and extended version of the original FRI MATLAB Toolbox. We present the initial version of the FRI toolbox based on OCTAVE language, which is open-source software (free license) available under the General Public License (GPL).

3.1. Preliminaries

This subsection provides some basic definitions of the complete fuzzy rule base and sparse rules. It also introduces the description of the interpolative reasoning concept.

3.1.1. Dense and Sparse fuzzy rule bases

Let us take into consideration the two numerical variables *X* and *Y*, which described on the universe *R* of real numbers, and *F* is a set in the fuzzy sets of *R*. We assume the fuzzy sets A_i in *F* are defined, $1 \le i \le n$, such that: $(A_1 \preccurlyeq A_2 ... \preccurlyeq A_i \preccurlyeq A_{i+1} ... \preccurlyeq A_n)$, for a given order $\preccurlyeq$ on *F*. We also suppose that we are given fuzzy sets B_i in *F*, $1 \le i \le n$, which also ordered according to $\preccurlyeq$.

According to the definitions in [12], [13], the fuzzy functions could be described by the fuzzy relations between the fuzzy sets of the inputs A_i and outputs B_i. The fuzzy rule base could characterize and represent based on this relation. The classical reasoning methods, such as Mamdani and Sugeno [1], [2] follow that relation, which require defining all the fuzzy rule base relations between the inputs and outputs, also to define the overlapping between them to obtain the desired conclusion. **Fig. 21** describes the complete fuzzy rule base for two dimensions' antecedents and consequents, the observations (*x1*) and (*x2*) match the fuzzy rules 1, 2, 4, and 5. Thus, the conclusion could be computed based on one of the classical fuzzy reasoning methods, like the Zadeh-Mamdani max-min Compositional Rule of Inference (CRI).

Regarding the sparse rule-base (incomplete rule-bases) systems, the fuzzy rules are of the type: (R_i): "*if X is* A_i *then Y is* B_i". The sparsity means, there is no overlapping between the observation and any of the fuzzy rules (do not cover the input space *F*), where there exist inputs A^* such that $(\exists_i / A_i \preccurlyeq A^* \preccurlyeq A_{i+1})$. A fuzzy interpolation method aims to provide a conclusion according to observation A^* and two adjacent rules R_i and R_{i+1} when $(A_i \preccurlyeq A^* \preccurlyeq A_{i+1})$.

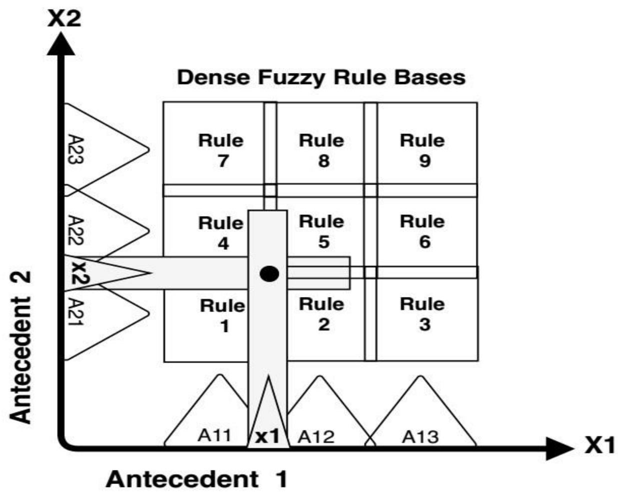

Fig. 21. *Complete Fuzzy Rule Base*

Fig. 22 describes the issue (incomplete fuzzy rule base), where the observations $x_{1.1}$ and $x_{1.2}$ refer to the first input (antecedent 1); the observations $x_{2.1}$ and $x_{2.2}$ refer to the second input (antecedent 2). These observations described two different types of issues in classical reasoning. The observations $x_{1.1}$ and $x_{2.1}$ do not match any rules of the rule-bases (missing rules), while, the observations $x_{1.2}$ and $x_{2.2}$ do not overlap with any fuzzy sets in the universe of discourse (gaps between fuzzy sets), there are no fuzzy values defined. Hence no overlapping rule can exist.

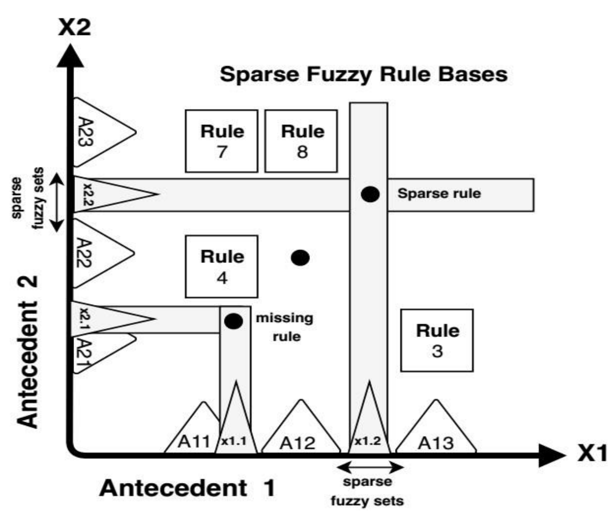

Fig. 22. *The Incomplete Fuzzy Rule Base (Sparse and no Overlapping Fuzzy Sets)*

3.1.2. Fuzzy rule interpolation notations

The fuzzy function definitions in [12], [13], the fuzzy space could be described by the mapping between antecedents and consequents fuzzy sets L^X and L^Y via ($f{:}\ L^X \rightarrow L^Y$), this leads

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to the main idea of the fuzzy rule interpolation methods, which is finding a suitable fuzzy interpolative function. These functions could produce a conclusion directly, even if the rule base is sparse, and there is no overlapping between the observation and any fuzzy rule base. Many of the Fuzzy Rule Interpolation (FRI) methods follow the notions in [3], [14], [15], which describe the relation between two fuzzy rule base and the observation, these fuzzy sets must be adjacent Convex and Normal (CNF) and partially ordered fuzzy sets, the ordering defined as (A_1, *is said to be "less than"* A_2), for all A_1, A_2 sets in a given fuzzy partition.

The ordering of the fuzzy set A_1 and A_2, denoted by A_1, A_2, *if* $\forall\alpha \in [0,1]$, the following condition hold: $(Inf(A_{1\alpha}) < Inf(A_{2\alpha}), Sup(A_{1\alpha}) < Sup(A_{2\alpha}))$, where the "*Inf*" denotes the infimum and "*Sup*" refers to the supremum of the $(A_{1\alpha})$, $(A_{2\alpha})$ fuzzy sets.

For simplicity, suppose that two fuzzy rules are given:

If X is A_1 *then Y is* B_1
If X is A_2 *then Y is* B_2

Where $A_1 \Rightarrow B_1$ *and* $A_2 \Rightarrow B_2$ describe the fuzzy rules, rules in a given rule base arranged concerning a partial ordering among the convex and normal fuzzy sets (CNF sets) of the antecedents, consequent, and observation. For the above two rules, this means that:

$$A_1 \prec A^* \prec A_2 \wedge B_1 \prec B_2 \tag{3.1}$$

Fig. 23 illustrates the simplest form to describe two flanking rules of the fuzzy sets, in which the shape of the membership functions remained restricted to trapezoidal, the figure shows the main points (variables) of the fuzzy sets to be applied for determining the conclusion in most FRI methods.

A_1 *and* A_2 refer to the fuzzy sets of the antecedents, B_1 *and* B_2 denote the consequent fuzzy sets. A^* denotes the new input (observation), B^* refers to the conclusion. The characteristic points of the trapezoidal membership function could be defined by four values (*LF, LC, RC, RF*), the (*LC and RC*) refer to the left and the right core, the (*LF and RF*) refer the left and the right flank. (RA_1, RA^*, RA_2) denote the center points of the fuzzy sets in antecedents side, and similarly, the (RB_1, RB^*, RB_2) denote the center points of the fuzzy sets in consequents side, (*fl, s2, r1*) and (*fu s1, r2*) denote the left and the right fuzziness for each fuzzy set, (*Ui, U'*) denotes the distance between the center points of the fuzzy sets.

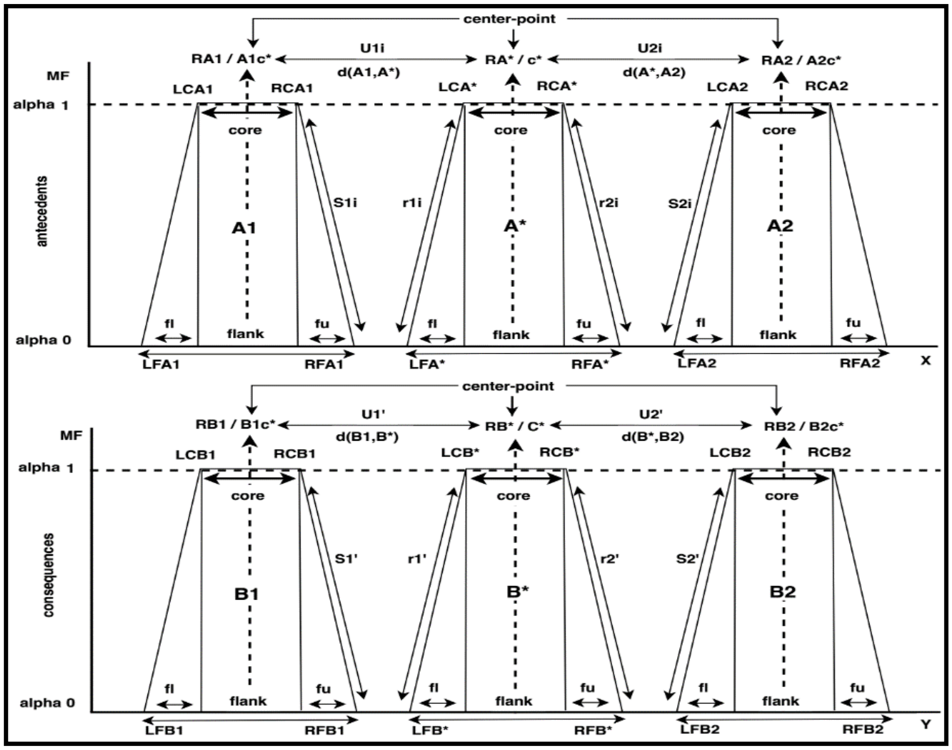

Fig. 23. *Fuzzy Interpolation with Trapezoidal Fuzzy Sets (The Antecedent Part and Observation) and (The Consequent Part and Conclusion) [32]*

3.2. Fuzzy Rule Interpolation Methods

Fuzzy rule interpolation methods classified into two groups. The first group obtains the conclusion in a single-step (directly), and the other group demands two-steps to compute the conclusion, using different algorithms in each step. This subsection presents an overview of some of the implemented FRI methods.

3.2.1. KH interpolation method

The first method proposed for FRI concept is called the KH (linear interpolation) method; this method published by Kóczy and Hirota [3]. Concerning the common general properties of the FRI methods suggested in [15], The KH rule interpolation method needs the following conditions to be satisfied: the fuzzy sets in both antecedents and consequents must be Convex and Normal (CNF) with bounded support, and at least a partial ordering must exist between fuzzy sets in the universes of discourse.

The conclusion in KH interpolation method produced directly based on the α-cuts of the observation and the fuzzy rule-base, it can be calculated by the fundamental equation of the KH-FRI (**Eq.(3.2)**), which based on the lower and upper fuzzy distances between fuzzy sets [16]. The upper and lower endpoints could use to calculate the distance between the conclusion and the consequent, which must be analogous to the upper and lower fuzzy distances between observation and antecedents.

$$d(A^*, A_1):d(A^*, A_2) = d(B^*, B_1):d(B^*, B_2) \quad (3.2)$$

Where (d) refers to the Euclidean distance that could use between the fuzzy sets (A_1, A_2) and (B_1, B_2).

The conclusion B^* in this method could be calculated based on the lower and upper fuzzy distances between the fuzzy sets of the antecedents, consequent, and observation. **Fig. 23** illustrates the main points (core and flank) of the trapezoidal fuzzy sets that could be used to compute the conclusion B^* as follows:

The right (core) can be calculated by **Eq.(3.3)**:

$$\mathrm{RCB}^* = \frac{d_1RC \times RCB_1 + d_2RC \times RCB_2}{d_1RC + d_2RC} \quad (3.3)$$

Where

$$d_1RC = \sqrt{\sum_{i=1}^{k}(RCA_i^* - RCA_{i1})^2}$$

$$d_2RC = \sqrt{\sum_{i=1}^{k}(RCA_{i2} - RCA_i^*)^2}$$

For the right (flank) can be calculated by **Eq.(3.4)**:

$$RFB^* = \frac{d_1RF \times RFB_1 + d_2RF \times RFB_2}{d_1RF + d_2RF} \quad (3.4)$$

$$d_1RF = \sqrt{\sum_{i=1}^{k}(RFA_i^* - RFA_{i1})^2}$$

$$d_2RF = \sqrt{\sum_{i=1}^{k}(RFA_{i2} - RFA_i^*)^2}$$

The left (core) and the (flank) can be obtained similarly to the above **Eq.(3.3)** and **Eq.(3.4)**.

The KH method was developed for a single dimension and multi-dimensional antecedent universes as appearing in the previous equations. The most significant advantage of the KH interpolation is its simplicity and its low computational complexity. However, the disadvantage of this method is abnormality conclusion could be appeared, it can be seen in some cases such as the cases in [17], [18], where the lower (left) end of the α-cut interval has a higher value than its upper (right) endpoint.

3.2.2. The KH Stabilized interpolation method

Many studies modified of the original KH method to improve the abnormal in conclusion, and to take more than two rules throughout the determination of the conclusion; the extended method was developed to handle and decrease the abnormality of the original KH method is called KH Stabilized, which was proposed by Tikk, .et.al. [5].

This method takes into account all flanking rules of observation in the calculation of the conclusion to the extent to the inverse of the distance of antecedents and observation. The universal approximation property holds if the distance function is raised to the power of the input's dimension.

The authors of [5] propose using formulas to calculate the upper and lower endpoints of α-cuts of the approximated consequence which contain the distance on the (n^{th}) power as shown via **Eq.(3.5)** and **Eq.(3.6)**:

$$\min B_{\alpha}^{*} = \frac{\sum_{i=1}^{m} \frac{\inf(B_{i\alpha})}{d_L^N(A_{\alpha}^{*}, A_{i\alpha})}}{\sum_{i=1}^{m} \frac{1}{d_L^N(A_{\alpha}^{*}, A_{i\alpha})}} \tag{3.5}$$

$$\max B_{\alpha}^{*} = \frac{\sum_{i=1}^{m} \frac{\sup(B_{i\alpha})}{d_U^N(A_{\alpha}^{*}, A_{i\alpha})}}{\sum_{i=1}^{m} \frac{1}{d_U^N(A_{\alpha}^{*}, A_{i\alpha})}} \tag{3.6}$$

The simplest of the KH Stabilized method is the linear interpolation of two rule-bases for the area between their antecedents. This method can also be applied if the observation position is located between two closest rules or hits outside rule-bases.

3.2.3. VKK interpolation method

VKK method was proposed by Vas, Kalmar, and Kóczy [4]. The main idea of VKK method is based on the center point and width ratio, the conclusion could be calculated by the center point and width ratio between the antecedent, consequent, and observation fuzzy sets.

The center point of the conclusion can be obtained by **Eq.(3.7)**, **Eq.(3.8)** and **Eq.(3.9)**:

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$$Center(B^*) = \frac{L.Center \times R.Center}{d(A_{i1\alpha}, A_{i2\alpha})} \quad (3.7)$$

$$L.Center = d(A^*_\alpha, A_{i2\alpha}) \times Center(B_{i1\alpha}) \quad (3.8)$$

$$R.Center = d(A_{i1\alpha}, A^*_\alpha) \times Center(B_{i2\alpha}) \quad (3.9)$$

where

$$Center(A_\alpha) = \frac{\inf(A_\alpha) + \sup(A_\alpha)}{2}$$

The width ratio of the conclusion can be calculated by **Eq.(3.10)**, **Eq.(3.11)** and **Eq.(3.12)**:

$$Width(B^*) = \frac{LfetWidth \times RightWidth}{d(A_{i1\alpha}, A_{i2\alpha}) \times WA^*} \quad (3.10)$$

$$LeftWidth = d(A^*_\alpha, A_{i2\alpha}) \times Width(B_{i1\alpha} / WA_{1i}) \quad (3.11)$$

$$RightWidth = d(A_{i1\alpha}, A^*_\alpha) \times Width(B_{i2\alpha} / WA_{2i}) \quad (3.12)$$

where

$$Width(A_\alpha) = \sup(A_\alpha) - \inf(A_\alpha)$$

The ($d(A_{i1\alpha}, A^*_\alpha)$, $d(A^*_\alpha, A_{i2\alpha})$ *and* $d(A_{i1\alpha}, A_{i2\alpha})$)) refer to the distance between antecedents fuzzy sets, the geometric average of the width values represented by (WA_{1i}), (WA_{2i}), and (WA^*) between the antecedents and observation.

The disadvantage of this method is the abnormal can appear in some cases. Nevertheless, the VKK method has a low complexity compared to the KH method due to the calculation of the conclusion directly through the center and the width of the fuzzy sets. It is also simple and used in several applications without complications.

3.2.4. MACI interpolation method

Another method of the FRI called the Modified α-Cut based Interpolation (MACI) method was proposed by Tikk and Baranyi [6]. The main idea of this method is based on the vector's description of the fuzzy sets for eliminating the abnormal problem in conclusion. The fuzzy set in this method could be described by two vectors space; it can represent the Left and the Right flank of the α-cut levels, where the abnormal consequent set is excluded.

The characteristic points are represented in vector description, which can be represented by the piecewise linear shape of the fuzzy sets, (a_{-1} and a_0) describe the left flank, and (a_0 and a_2) represent the right flank; also a_0 refers to the reference point of the fuzzy set, the Cartesian axes can be represented by Z_0, Z_1 as shown in **Fig. 24**.

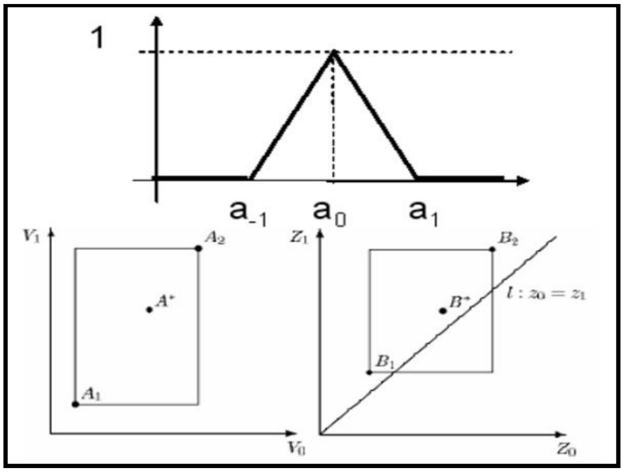

Fig. 24. *The Vectors Description Input and Output Fuzzy Sets [6]*

The conclusion in this method could be determined by the transformation of the current characteristic points to a new Cartesian to calculate the conclusion, then transforming back to the original Cartesian to show the result that could be computed by the following **Eq.(3.13)**, **Eq.(3.14)** and **Eq.(3.15)**:

The new Cartesian can be calculated by the vector form:

$$b = [b_0, b_1] \text{ and } b' = [b_0', b_1'] \tag{3.13}$$

$$b_0' = b_0 . \sqrt{2} \text{ and } b_1' = b_0 - b_1 \tag{3.14}$$

The matrix can represent vector description as:

$$b' = bT \tag{3.15}$$

where

$$T = \begin{bmatrix} \sqrt{2} & 0 \\ -1 & 1 \end{bmatrix}$$

The MACI method concentrates on the characteristic points of the fuzzy set (A_1, A^*, and A_2) and the consequents (B_1 and B_2). It can be described by vectors that involve computing the center point of the conclusion RB^* as shown in **Fig. 23**. The conclusion could be calculated by **Eq.(3.16)** as follows:

$$RB^* = (1-\lambda_{core}).RB_1 + \lambda_{core}.RB_2 \tag{3.16}$$

where

$$\lambda_{core} = \frac{\sqrt{\sum_{i=1}^{k}(RA_i^* - RA_{i1})^2}}{\sqrt{\sum_{i=1}^{k}(RA_{i2} - RA_{i1})^2}}$$

Where, RA^*, RA_1, *and* RA_2 denote the reference point of the observation and antecedents fuzzy sets. After computing the conclusion could be transformed back to the original Cartesian by the vector, applying **Eq.(3.17)**, **Eq.(3.18)** and **Eq.(3.19)**:

$$b_0^* = b_0^{*'}.\sqrt{2} \tag{3.17}$$

$$b_1^* = b_1^{*'} + b_0^* = b_1^{*'} + (b_0^{*'} / \sqrt{2}) \tag{3.18}$$

$$b^* = b^{*'}.T^{-1} \tag{3.19}$$

where

$$T = \begin{bmatrix} 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1 \end{bmatrix}$$

For more details of MACI function can be found in [19], [20]. The main advantage of the MACI method; the conclusion is always producing a convex and normal fuzzy set, and it can also apply multi-dimensional antecedents [6]. However, the disadvantage of this method (in some instances) does not keep a piecewise linearity of the membership functions.

3.2.5. CRF interpolation method

This method was proposed to modify the fuzziness term and to improve α-cut levels. The main idea of this method was introduced in [21], which was called GK method; also, the modified version of the GK called the KHG method was published by Kóczy, Hirota, and Gedeon in [7]. The current modified version is called the Conservation Relative Fuzziness (CRF), which follows the fundamental Equation (*FEFRI*) (**Eq.(3.2)**). CRF method aims to obtain the conclusion based on determining the core and fuzziness of antecedents, consequents, and observation fuzzy sets, the core c^* could be described by (A_1c^*, A_2c^*, A^*c^*) and (B_1c^*, B_2c^*) as shown in **Fig. 23**, using the distances between the antecedents and observation ($d(A^*, A_1)$ and $d(A_2, A^*)$), and between the consequents fuzzy sets $d(B_1, B_2)$.

Besides, the fuzziness of the conclusion could be determined by calculating variables (A_1fU, A^*fL) that must have the same fuzziness of the (B_1fU, B^*fL), similarly, the fuzziness between (A^*fU, A_2fL) and (B^*fU, B_2fL) as shown in **Fig. 23**.

The core of the conclusion C^* can be calculated by **Eq.(3.20)**:

$$C^* = c^* \times \frac{d_1(B_1, B_2)}{d_1(A_1, A_2)} \tag{3.20}$$

Where c^* denotes the core of the observation, and $d1$ denotes the distance between A_1 and A_2 which can be calculated as follows:

$$d_1 = (A_1, A_2) = A_2c^* - A_1c^*$$
$$d_1 = (B_1, B_2) = B_2c^* - B_1c^*$$

The general fundamental **Eq.(3.2)** can be applied to determine the distance between the current fuzzy sets through **Eq.(3.21)**:

$$\frac{d_1(A_1, A^*)}{d_1(A^*, A_2)} = \frac{d_1(B_1, B^*)}{d_1(B^*, B_2)} \tag{3.21}$$

So, **Eq.(3.22)** and **Eq.(3.23)** can be used to calculate the core of the conclusion using distance as follows:

$$d_1(B_1, B^*) = \frac{d_1(A_1, A^*) \times d_1(B_1, B_2)}{d_1(A_1, A_2)} \tag{3.22}$$

Similarity,

$$d_1(B^*, B_2) = \frac{d_1(A^*, A_2) \times d_1(B_1, B_2)}{d_1(A_1, A_2)} \tag{3.23}$$

The distance between the fuzzy sets (e.g., antecedent fuzzy sets A_1, A_2) can be computed as the following **Eq.(3.24)** :

$$d_1(A_1, A_2) = \sqrt{\frac{(A_{2c*} - A_{1c*})^2}{range^2}} \tag{3.24}$$

The fuzziness of the conclusion can be determined by the left and the right flanks using the current fuzzy sets as follows by **Eq.(3.25)** and **Eq.(3.26)**:

$$B^*_{fL} = A^*_{fL} \times \frac{B_{1fU}}{A_{1fU}} \tag{3.25}$$

$$B^*_{fU} = A^*_{fU} \times \frac{B_{2fL}}{A_{2fL}} \tag{3.26}$$

Equations in [7] could be used to compute $(A^*fL, A^*fU, A_1fU, A_2fL, B_1fU, B_2fL)$, which based on the calculation of the (inf) and (sup) of the current fuzzy sets. The previous **Eq.(3.20)** -

Eq.(3.26) of the CRF method were introduced to be applied by single-dimensional input, and also, it can apply in multi-dimensional input using the expression in [7].

The advantage of the CRF method is that the flanks used to define the conclusion, the CRF method can be applied as arbitrarily on fuzzy set shapes. Additionally, the observation position must be surrounding two rule-based to get a conclusion.

3.2.6. IMUL interpolation method

This method was proposed by Wong, Gedeon, and Tikk [8], the IMUL introduced to avoid the abnormal conclusion and improve the multidimensional α-cut (levels). IMUL method was presented to combine the features of the MACI method [6] and Conservation of Relative Fuzziness (CRF) method [7].

The IMUL method applied the vector description; it can describe the characteristics points of the fuzzy sets through advantageous the transformation feature of the MACI method and representing the fuzziness of the input and output by the CRF method. As shown in **Fig. 23,** the conclusion could be calculated between the characteristic points of the antecedent fuzzy sets, which are neighboring to the observation.

The conclusion in the IMUL method is based on calculating the reference point (RB^*) and the left / right core (LCB^*, RCB^*); the reference point could be computed by **Eq.(3.16)**. The left and right core can be calculated through the following **Eq.(3.27)** and **Eq.(3.28)**:

The right core:

$$RCB^* = (1-\lambda_{right}).RCB_1 + \lambda_{right}.RCB_2 + (\lambda_{core} - \lambda_{right}).(RB_2 + RB_1) \tag{3.27}$$

where

$$\lambda_{right} = \frac{\sqrt{\sum_{i=1}^{k}(RCA_i^* - RCA_{i1})^2}}{\sqrt{\sum_{i=1}^{k}(RCA_{i2} - RCA_{i1})^2}}$$

The left core:

$$LCB^* = (1-\lambda_{left}).LCB_1 + \lambda_{left}.LCB_2 + (\lambda_{core} - \lambda_{left}).(RB_2 + RB_1) \tag{3.28}$$

where

$$\lambda_{left} = \frac{\sqrt{\sum_{i=1}^{k}(LCA_i^* - LCA_{i1})^2}}{\sqrt{\sum_{i=1}^{k}(LCA_{i2} - LCA_{i1})^2}}$$

The conclusion flanks (LFB^*, RFB^*) can be computed by following **Eq.(3.29)**:

The left flank:

$$LFB^{*} = LCB^{*} - r_k (1 + \left| \frac{S'}{U'} - \frac{S}{U} \right|) \qquad (3.29)$$

The LFB^* denotes the left flank fuzziness of conclusion B^*, and the LCB^* refers to the left core; the right flank can be calculated in the same way. The variables (*r, s, u, s', u'*) are used to determine the fuzziness between the fuzzy sets and to calculate the conclusion flank ([8], [19]).

One of the benefits of using the IMUL method is that the conclusion can be obtained by computing core and fuzziness focusing on the information of the consequents (outputs) and the information of the antecedents' fuzzy sets that are given correct results. Moreover, the IMUL method can be applied on a single dimension and also in multi-dimensional inputs space (see the examples in [8]).

3.2.7. GM interpolation method

The first method in the second group of the interpolation methods that demanded two-steps to get the conclusion is called the General Method (GM) method. The GM was published by Baranyi et al. [9], two algorithms could be used to determine the conclusion in this method. The first one is based on the fuzzy relation. The second one is based on the semantics of the relations. The GM method adopts the characterization of the position fuzzy sets to determine the reference points (core). Thus, the distance between the observation and antecedents fuzzy sets can be calculated based on the reference points via **Eq.(3.30)** instead of using the interpolating α-cut levels.

$$d(A_1,A_2) = | RP(A_2) - RP(A_1) | \qquad (3.30)$$

Where A_1 and A_2 are the fuzzy sets, the reference point is *RP*, and *d* denotes the distance of the sets. The conclusion (interpolated) can be obtained by applying the following primary two steps:

The first step is to generate a new interpolated rule $R^i: A^i \rightarrow B^i$, which is between rules R_1 and R_2 via **Eq.(3.31)**, the position of the new rule is the same position of the observation, so to produce the new rule will be used each fuzzy set of the antecedents, which must be identical with the reference point of the observation fuzzy set in the corresponding dimension.

$$R^i = f^{Interpolation}(R_1, R_2) \qquad (3.31)$$

This step divided into three stages:

1. In the first stage, a set interpolation technique, will help to determine the antecedent set shapes of the interpolated rule.

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2. In the second stage, the reference point of the conclusion could be calculated by the reference points of the observation and the consequent sets, taking into consideration the adjacent rule bases, for example, using the fundamental equation of the fuzzy rule interpolation (FEFRI) (**Eq.(3.2)**).
3. In the third stage, the shapes of the consequent sets could be determined by the interpolated rule using the same set interpolation technique as (stage 1), as shown in **Fig. 25**.

Many techniques were proposed for this step of the set interpolation technique (e.g., SCM, FPL, Etc.). The Solid Cutting Technique (SCM) is introduced for this step, the main idea of this technique is that all the associated sets are rotated by 90˚ about a vertical axis passed through their reference point. By connecting the similar points of antecedents and consequents, two solids can be constructed: one in the input and one in the output dimension (**Fig. 25**), where the solid was created in an input dimension.

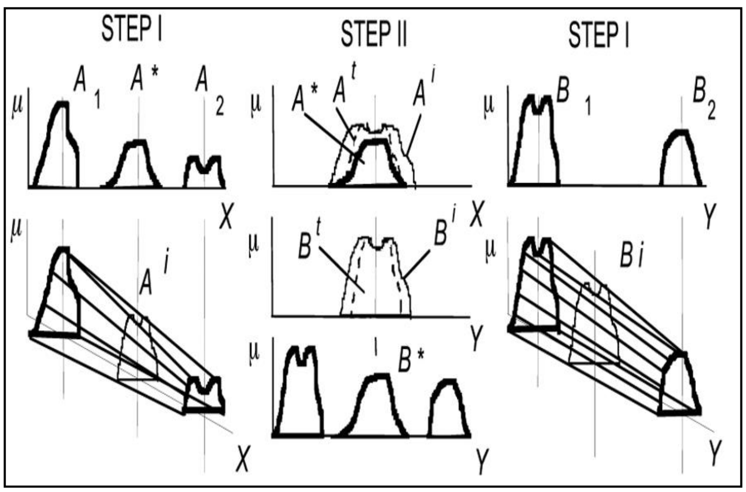

Fig. 25. *The Main Steps of the GM Method [9]*

The second step in the GM method is the new rule could be specified as a part of the extended rule of the approximate conclusion, as a conclusion of the inference method is defined by determining this rule. In many instances, there is no identical similarity between the rule and the observation part, for this purpose, many techniques are used to handle the mismatch by either the Transformation of the Fuzzy Relation (TFR) technique or by Fixed Point Law (FPL).

This step could be divided into two stages:

The first stage, the TFR technique could be applied, where the interrelation function [9] is generated between the observation (A^*) and the antecedent (A^i) set, there is mapping between observation (A^*) and antecedent (A^i) by the endpoints of the support and reference point (*RP*), as shown in **Fig. 26**, the interrelation area can be represented by the endpoints of the support sets. The purpose of the first phase is to improve the proportion of the area of interrelation

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mapping between (A^i *and* B^i) sets; it can correspond with the support of the observation and the horizontal side of the square. Hence, the support of the conversion set (A^t) is the same support of the (A^*), the membership in both cases (A^t, A^*) is the same as its interrelated point in the Antecedents part.

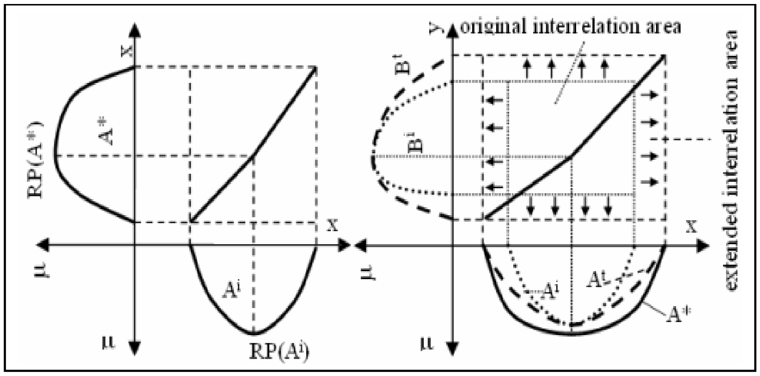

Fig. 26. *The Interrelation Functions [9]*

The second stage, the Fixed Point Law (FPL) technique can be used, where an interrelation function is created between observation (A^*) and the transformed antecedents sets (A^t), it is also used to calculate the difference between the membership values for each interrelated point set, this difference can be applied to determine the approximate conclusion from the transformed consequent sets (B^t) that will take into consideration the interrelation between transformed (A^t) and transformed (B^t) [9].

The main advantage of this method (GM) is to avoid the abnormal fuzzy conclusion; there is no restriction to CNF sets and preserving normality; it preserves linearity and is compatible with the rule base, finally, it investigates the monotonicity and the continuity.

3.2.8. FRIPOC interpolation method

This method was proposed by Johanyák and Kovács [10], which is called Fuzzy Rule Interpolation based on Polar Cuts (FRIPOC), FRIPOC method is based on the reasoning method by the concept of the linguistic term shifting and polar cut, it is appropriate in case of sparse and dense rule bases. The general formula can be described to show the reference point, which is specified to calculate the interpolated of the antecedents $RP(A^i_j)$ and the consequent $RP(B^i_l)$ sets, which could be calculated by **Eq.(3.32)**.

$$RPB_l^i = f(RP(A_1^i), RP(A_2^i), .., RP(A_j^i), .., RP(A_{na}^i),) \tag{3.32}$$

This method is based on the position of the fuzzy sets, which is characterized by a reference point during the calculations; the reference point RP(B^i_l) can be determined by several techniques **Fig. 27**, the reference point of this technique determined by **Eq.(3.33)** and **Eq.(3.34)**. The FRIPOC method mostly follows the GM method [9], where the conclusion can be done by applying two steps: the first step is to define the new rule based on the position of the antecedents part that describes the observation in each dimension, this means the reference point of the observation and antecedents set are identical.

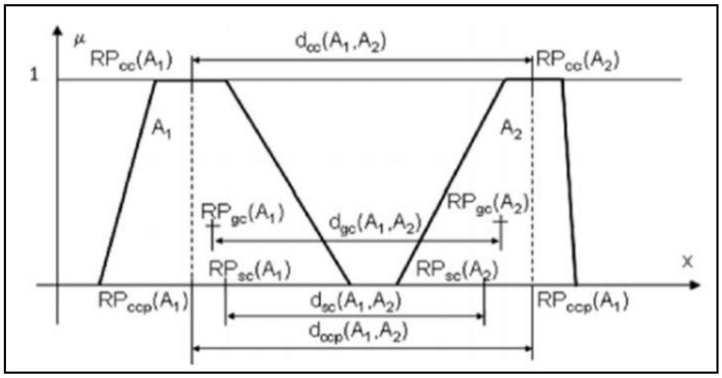

Fig. 27. *Choices for the Reference Point and the Associated Set Distances [10]*

$$RP(B_l^i) = \frac{\sum_{j=1}^{N} RP(B_{lj}).s_j}{\sum_{j=1}^{N} s_j} \tag{3.33}$$

$$s_j = \frac{1}{d(RA^i, RA_j)^2} = \frac{1}{\sum_{k=1}^{na} d(RP(A_k^i), RP(A_{jk}))^2} \tag{3.34}$$

Where $RP(B^i{}_l)$ is the *RP* of the consequent sets, s_j denotes to weight attached to the rule, l refers to the number of dimensions, (N) denotes the number of the rules, *(j)* refers to the actual rule, RA^i and RA_j denote the antecedent rule [10].

The new rule determined by two steps: The first step is described by three stages as follows: 1) the fuzzy sets of the antecedents are estimated using the set interpolation technique Fuzzy SEt interpolAtion Technique bases on Polar cut (FEAT-p) that is independently in each antecedent dimension, the main purpose of this technique is that the whole sets of the partition are shifted horizontally into the reference point of the observation, i.e., their reference points are identical with the interpolation point. 2) the new fuzzy set is determined based on the polar cut, where the fuzzy set can be specified using the polar distance of each polar cut level as a weighted mean of the similar polar distances of the forecasted identified sets. 3) the fuzzy set will determine the consequent by FEAT-p technique in the same way as (first stage). Thus, the new fuzzy set can be calculated by following the formula shown in **Eq.(3.35)**.

$$\rho(A_{j\theta}^i) = \begin{cases} \dfrac{\sum_{k=1}^{nj} w_{jk.\rho}(A_{jk\theta})}{\sum_{k=1}^{nj} w_{jk}} & , d(A_j^*, A_{jk}) > 0 \\ \rho(A_{jk\theta}) & d(A_j^*, A_{jk}) = 0, k = 1..n_j \end{cases} \tag{3.35}$$

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The second step in the FRIPOC method defines the conclusion by exciting the new rule based on the Single Rule Reasoning based on polar cuts (SURE-p) technique [10]. The reference point of the interpolated conclusion and the consequent set are identical to the new rule in the current dimension. **Fig. 28** describes the distance of the polar that can be calculated based on each polar level; the conclusion can compute by the modified consequents of the interpolated rule using the average differences, the technique of correction and control could be used to guarantee the efficacy of the new fuzzy set.

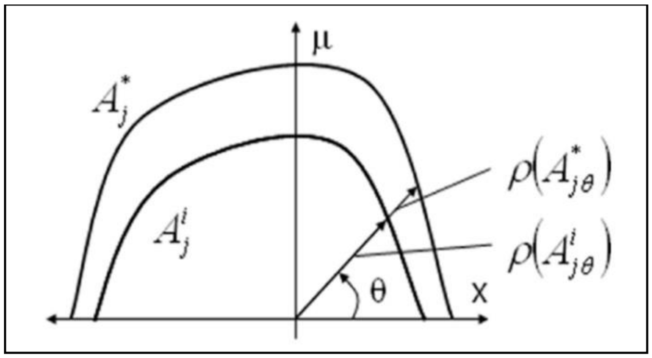

Fig. 28. *Polar Distances Utilized for the Estimation of the Relative Difference [11]*

The main benefits of the FRIPOC method are comprehensibility, the ability to applicability in subnormal cases, and can be applied if there are no rules surrounding the observation (extrapolation).

3.2.9. LESFRI interpolation method

This method follows the GM method by computing the conclusion based on two steps; this method is called LEast Squares based Fuzzy Rule Interpolation (LESFRI), which was proposed by Johanyák and Kovács [11].

The main idea of LESFRI method is the conservation of the weighted average differences measured on the antecedent part, where these modifications could be applied on the consequent side, in which the results usually could be as a set of characteristic points that will not fit with the default shape type of the partition. Therefore, the LESFRI method could be used in order to find the breakpoints of a satisfactory conclusion.

The LESFRI method is based on two-step:

The first step aims to define the interpolation point of the new fuzzy set, which can be achieved by three stages as follows [11]:

1. The FEAT-LS technique is used to calculate the antecedent sets for each dimension, where this technique aims to generate a new fuzzy set based on the interpolation points of the

fuzzy partitions; thus, all the sets of the partition are shifted horizontally in order to reach the coincidence between their reference points and the interpolation point by **Eq.(3.36)**.

$$Q_j^L = \sum_{l=1}^{n} w_{lj}.(X_{lj}^L - X_j^L)^2 \tag{3.36}$$

where

$$w_j = \frac{1}{d(A_j, A^i)^p}$$

2. The position of the consequent fuzzy sets can be determined for each consequent dimension of the new rule by utilizing a crisp interpolation method by **Eq.(3.37)**.

$$s_l = \frac{1}{d(RA^i, RA_j)^2} = \frac{1}{\sum_{k=1}^{na} d(RP(A_j^i), RP(A_{lj}))^2} \tag{3.37}$$

3. The characteristic points of the new fuzzy sets shapes are defined by the weighted least squares by taking into consideration the similar characteristic points of the overlapped sets, which could be used to estimate the conclusion using the observation and the new rule.

The second step aims to produce the conclusion based on the new rule because the points of the rule do not fit ideally with the observation in each input dimension. The SURE-LS method (as a single rule reasoning method) was proposed for this purpose based on the α-cut approach. Consequently, all the current antecedent dimensions and consequent fuzzy sets could be described by the breakpoint α-levels to calculate the conclusion, it must be done independently to the left and right flanks of the fuzzy sets. Additionally, the weighted average of the distances between the endpoints α-cuts of the rule antecedent and the observation set could be calculated to each side for each level. The advantages of this method are its capability to produce new linguistic terms that fit into the regularity of the original partitions, and its low computational complexity, where it can be applied in case of the interpolation and extrapolation.

3.2.10. Scale and Move interpolation method

The scale and move transformation-based method was produced by Huang and Shen [22]; it follows the interpolation concept to handle the sparse fuzzy rule-bases. The scale and move method provides the capabilities to work with different fuzzy membership functions types such as (Triangular, Trapezoidal).

The scale and move method is based on the Centre of Gravity (COG) of the membership functions, as shown in **Fig. 29**. It creates a new central rule via two neighboring rule-bases that are surrounding the observation.

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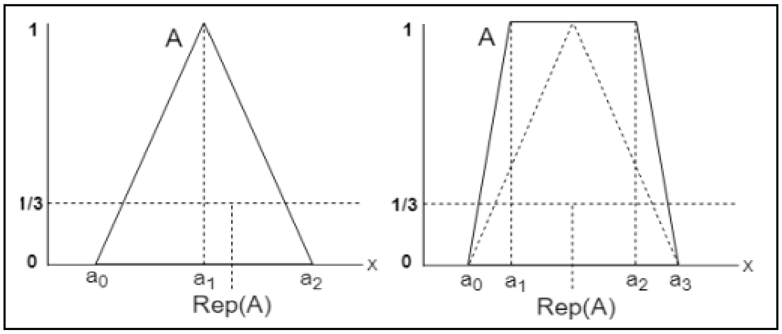

Fig. 29. *Representative Value of a Triangular and Trapezoid Fuzzy Sets [22]*

This ScaleMove method takes two-steps to obtain the conclusion, the first step is to produce a new central rule *(A` → B`)* is produced within the existing surrounding rule between observation $(A^{*}: A_1 \rightarrow B_1, A_2 \rightarrow B_2)$ by applying **Eq.(3.38)**:

$$\lambda_{REP} = \frac{d(REP(A_1), REP(A^{*}))}{d(REP(A_1), REP(A_2))} \quad (3.38)$$

Where $d(Rep(A_1); Rep(A_2))$ represents the distance between two fuzzy sets A_1 and A_2. $Rep(A_1)$ refers to the center of gravity for A_1 [22].

The new rule-base ($A' \rightarrow B'$) can be calculated by **Eq.(3.39)** and **Eq.(3.40)**:

$$A' = (1-\lambda_{REP})A_1 + \lambda_{REP}A_2 \quad (3.39)$$

$$a_0' = (1-\lambda_{REP})a_{10} + \lambda_{REP}a_{20}$$

$$a_1' = (1-\lambda_{REP})a_{11} + \lambda_{REP}a_{21}$$

$$a_2' = (1-\lambda_{REP})a_{12} + \lambda_{REP}a_{22}$$

$$B' = (1-\lambda_{REP})B_1 + \lambda_{REP}B_2 \quad (3.40)$$

$$b_0' = (1-\lambda_{REP})b_{10} + \lambda_{REP}b_{20}$$

$$b_1' = (1-\lambda_{REP})b_{11} + \lambda_{REP}b_{21}$$

$$b_2' = (1-\lambda_{REP})b_{12} + \lambda_{REP}b_{22}$$

The degree of similarity between A' *and* A^{*} is set, it is natural to require the consequent part B' *and* B^{*}, which achieve the same similarity degree as follows:

The more similar X to A'*; the more similar Y to* B'

The second step is to calculate the A' similarity degree between fuzzy sets *(A' and A^*)* that transform (*B' to B^*)* with the desired degree of similarity by the scale and move transformation.

The *Scale* transformation aims to change the support value of the membership function while keeping its representative value and shape; the *Move* transformation aims to transfer the support of the membership function with the keep of its representative.

The advantages of scale and move method that it can handle multiple antecedent variables with simple computation. It guarantees the normality and convexity of the conclusion fuzzy set. It offers the capability to handle the extrapolation issue directly [23]. It preserves the piecewise linearity for interpolations involving arbitrary polygonal fuzzy sets, and it uses various definitions for representative values.

3.3. Applications of Fuzzy Interpolation

Fuzzy interpolation systems have been successfully applied to many real-world problems. In the following, we present most of the fuzzy interpolation applications.

3.3.1. FRI with Truck backer-upper control

Backing a trailer truck to a loading dock is a challenging task for all yet the most skilled truck drivers. Because of the difficulties, this challenge has been utilized as a control benchmark problem with various solutions proposed [104], [105], [106]. For example, an artificial neural network has been applied to this problem, but a large amount of training data is required. An adaptive fuzzy control system was also suggested for this problem, but the creation of the rule base is computationally expensive. Another solution combines empirical knowledge and data. A combined fuzzy rule base is generated by joining the previously generated rules and linguistic rules.

Fuzzy interpolation system has also been applied to the trailer truck backer-upper problem [107] to further reduce system complexity. The problem can be formally formulated as $\theta=f(x, y, \emptyset)$. Variables x and y represent the coordinate values corresponding to horizontal and vertical axes; $\emptyset$ refers to the azimuth angle between the truck's onward direction and the horizontal axis; and θ is the steering angle of the truck. Given that enough clearance is present between the truck and loading lock in most cases, variable y can be safely omitted and hence results in a simplified formula to $\theta=f(x, \emptyset)$. By evenly partitioning each variable domain into three fuzzy sets, nine (i.e., 3*3) fuzzy rules were generated using FISMAT [108] and each of which is denoted as *IF x is A AND $\emptyset$ is B THEN θ is C*, where *A, B* and *C* are three fuzzy values. Noting that domain partitions appear to be symmetrical in some sense, the three rules that are flanked by other rule pairs were removed from the rule base resulting a more compact rule base with only six fuzzy rules.

3.3.2. FRI with Heating system control

Reducing domestic energy waste contributes to realizing the legal binding purpose in the UK that CO2 emissions need to be reduced by at least 34% below base year (1990) levels by 2020. Space heating consumes about 60% of the home energy consumption, and the home Electricity Survey has reported it from GOV.UK, that 23% of homeowners leave the heating on while going out. To reduce the waste of heating unoccupied homes, several of sensor-based and programmable controllers for the central heating system have been developed, which can successfully switch off the home heating systems when a property is unoccupied. Nevertheless, these systems cannot automatically preheat the homes before occupants return without hand-operated inputs or leaving the heating on uselessly for a longer time, which has limited the wide application of such devices.

To address this limitation, the authors in [109] presented a smart home heating controller, which can control the heating system to preheat the property before home users going home. The controller is developed by adapting fuzzy rule interpolation, supported by location information through portable devices. The system first predicts the time before home users go home; then the time to preheat home is approximated. If the predicted time to going home is not greater than the time to preheat home, the heating system will be turned on. As shown in the demonstrative example in [109], the proposed system can automatically present a solution to preheat the home when there is a need, but not leaving the heating system on all the time resulting in energy loss. The work can be further improved despite its hopeful results. Pre-defined rule base with no learning capability may not be perfectly suitable for all users' situations, which may be solved by allowing dynamic and adaptive rule base generation based on users personal data. It is possible that the home heating system does not switch on when the user comes back or the heating system has been switched on too early, as the proposed system does not have any error correction function to solve such issues. Adaptive fuzzy interpolation approach could be used to track the error back and modify the faulty part when incorrect results are generated.

3.3.3. FRI as a model for student result prediction

The author in [110] introduced reports on the creation of a fuzzy model that can predict the exam results of students based on their previous university achievements. This type of prediction can never tell precisely the exam results in advance because the previous academic life of the students does not fully determine in advance the exam results. Nevertheless, an excellent sufficient estimation can present great help for the university timetable and resource allocation planning. In the case of this project, the root means squared error expressed in percentage of the output range was less than 13% at the end of the tuning process in the case of all datasets that give an adequate level of information for planning the number of student groups and laboratory classes in the next semester in the case of the ASP.Net Programming course that follows the examined Visual Programming course. The developed fuzzy model contains only 28 rules mainly because not all the value combinations of the 9 input variables could have experimented in practice. Accordingly, a fuzzy rule interpolation-based inference technique had to be adapted. The results of this project proved that the presented approach is suitable for higher education applications for the prediction of students' exam results in their third or higher years of studies.

3.3.4. FRI with SNMP-MIB for Emerging Network Abnormality

The authors in [111] presented the benefits of the FRI in the Intrusion Detection Systems (IDS) application area, for the design and implementation of the detection mechanism for Distributed Denial of Service (DDOS) attacks. The FRI-IDS application's performance was compared to other common classification algorithms (support vector machine, neural network, random forest, and decision tree) used for detecting DDOS attacks on the same open-source test-bed environment. According to the results, the overall detection rate of the FRI-IDS is in pair with other methods. Consequently, the FRI inference system could be a suitable approach to be implemented as a detection mechanism for IDS; as it effectively decreases the false positive rate value.

3.3.5. FRI with Face Detection and Expression Recognition

In [112], an expression recognition system utilizing the Fuzzy Rule Interpolation (FRI) technique to classify and recognize 7 categories of facial expression was developed. The experimentation result proves that FRI technique is a promising method and can perform better. The technique offers important tools that will enable expression recognition and the ability to make a conclusion in situations where there is a missing or sparse rule in the rule base. Face detection and expression recognition from facial images is a useful area of research that has its application in computer vision, human-computer interaction, robotics and security systems. The study aimed to classify facial expression by first detecting a face (using the Viola-Jones Algorithm), extracting the features from the detected face with LBP and recognizing the expression using a fuzzy rule interpolation technique. The seven different facial expressions from different image subjects were obtained from the Extended Cohn Kanade dataset and analyzed.

3.3.6. FRI with Detecting Slow Port Scan

The authors in [113] introduced a novel strategy for detecting port scan attacks. The proposed strategy was designed and constructed using a fuzzy rule interpolation. The FRI-based detection strategy's inference engine was performed using Fuzzy Rule Interpolation based on the Polar Cuts (FRIPOC) method. The sparse fuzzy rules were generated based on expert knowledge, the range values of the input parameters during the experiments' four phases, and the relationship between the input parameters and the number of attacker clients. The conducted experiments reflect the proposed FRI based detection strategy's ability to effectively detect the very slow and slow port scans based solely on the sparse fuzzy rules. The FRI-based detection strategy's output responses were compared with SNORT, and the results reflected that the proposed detection approach was successful in detecting the very slow port scan attack in instances where the SNORT did not render any alert. Moreover, the FRI-based detection approach presented additional information, such as the level of port scan attack, instead of a binary alert.

3.3.7. FRI with Intrusion Detection Mechanism

The authors in [114] investigated the capabilities to use the FRI methods in the IDS application area. This investigation is practiced by implementing the FRI-IDS model as a detection mechanism for DDOS attack. The FRI-IDS model was constructed using the sparse fuzzy model identification. The fuzzy rules of FRI-IDS model were generated and optimized using RBE-DSS method. According to the example in [114], using an open-source DDOS dataset, the results of FRI-IDS model were compared with other literature's results, which they applied different algorithms to detect the DDOS attacks using the same testbed environment. The implemented experiments have demonstrated that the FRI-IDS model obtained an accepted detection rate. It has reduced effectively the false positive rate value, which decreased a large amount of IDS alerts.

Additionally, the FRI-IDS model can serve the interpolated conclusions even in case if some observations are not covered directly by fuzzy rules. Consequently, the FRI-IDS model could be a suitable approach for detecting intrusions if it is implemented as a detection mechanism. It is characterized by offering the capability to present the detection level of intrusion and permits the attack alert generation in case of a lack of information and definition of an existing knowledge base.

The authors in [115] presented a data-driven network intrusion detection system by applying the recently proposed TSKinterpolation approach. The experiment results using the benchmark data set KDD-99 described that the proposed system is able to generate security alerts for known attack types successfully, and to detect the unknown types of treats with success thanks to its good generalization capability. This work can be enhanced by employing the recently proposed rule base generation approach to generate a sparse rule base directly from very complex training data sets, and rule base adaptation approach to allow the rule base to be adapted and enhanced along with the operation of the IDS system. Also, the proposed work is developed using TSKinterpolation, and it is worthwhile to investigate how the proposed system may be developed by employing other fuzzy interpolation approaches with Mamdani-style rules bases.

3.3.8. FRI with behaviour-based control structures

The authors in [116] introduced an interpolation based fuzzy reasoning method, which could be implemented to be simple and quick enough to fit the requirements of behaviour-based

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control structures in real-time direct fuzzy logic control systems. The suggested approximate fuzzy reasoning method based on KH interpolation in the vague environment of the fuzzy rule base gives an efficient way of designing direct fuzzy logic control applications. The lack of fuzziness in the conclusion is a disadvantage of the proposed method, but it does not influence in common applications where the next step after the fuzzy reasoning is the defuzzification. To prove the efficiency of the interpolation-based fuzzy reasoning in behaviour-based control, a fuzzy behaviour-based control structure based on the fusion of different known behaviors in the function of their actual necessities approximated by a fuzzy automaton is also introduced in this paper briefly.

The implementation of interpolation-based fuzzy reasoning methods in behaviour-based control structures simplifies the task of fuzzy rule base creation. Since the rule base of a fuzzy interpolation-based controller, is not necessarily complete, it could contain the most significant fuzzy rules only without risking the chance of having no conclusion for some of the observations. In other words, during the construction of the fuzzy rule base, it is enough to concentrate on the cardinal actions; the "filling" rules (rules could be deduced from the others) could be deliberately omitted. Thus, compared to the classical h z y compositional rule of inference, the number of the fuzzy rules needed to be handled during the design process could be dramatically reduced.

3.3.9. FRI with Hotels Location Selection

The authors in [117] presented a hierarchical fuzzy decision model for selecting tourist hotel location. it has addressed the applicability and usefulness of the proposed BFRI approach for hotel location selection assessment and decision-making support. Fuzzy reasoning based HLS assessment systems offers significant potential for providing decision-making support in hotel location selection.

Systematic analysis of a hotel location selection assessment framework has been given. A four-layered analytical process with a detailed description for HLC is considered. It presents such an integrated approach capable of dealing with dynamic and insufficient information in the HLC process. In particular, the hierarchical system implementing the proposed technique can predict the ideal solution on different segments of focused attention and help identify hidden variables that may be useful during the decision support process (by performing reverse inference).

3.3.10. FRI with Detection of IoT-botnet attacks

The authors in [118] introduced a novel approach for detecting IoT-Botnet attacks within the IoT smart environments by adapting the LEast Squares based Fuzzy Rule Interpolation (LESFRI) method. The proposed approach was designed and implemented using a sparse fuzzy model identification (RBE-DSS). The proposed approach eliminates the need for creating a complete fuzzy rule base to detect the IoT-Botnet attack. Therefore, one of the distinctive

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advantages of the proposed approach, its ability to generate the required IDS alert even in case if the attack knowledge-base is incomplete. Furthermore, the proposed approach effectively smoothes the boundary between normal and attack traffics because of its fuzzy-nature. The Rule Base Extension using the Default Set Shapes (RBE-DSS) method generated the sparse attack rule-base. The proposed approach was tested and evaluated using a recent open-source benchmark IoT dataset. The experiments applied on an IoT-Botnet attack benchmark dataset were demonstrated, that the proposed approach could achieve an acceptable detection rate. Moreover, it was able to detect the IoT-Botnet attack in cases successfully not covered directly by any of the fuzzy rule antecedents.

3.4. Fuzzy Rule Interpolation Toolboxes

The goal of this subsection are: Firstly, to introduce a brief description of the refreshed and extended version of the MATLAB FRI Toolbox. Secondly, to present the initial version of the FRI toolbox under the OCTAVE environment.

3.4.1. Fuzzy Rule Interpolation MATLAB Toolbox

The FRI toolbox was developed by Z.C. Johanyák, .et. al. [24] and implemented in the MATLAB environment. The main goal of the FRI toolbox is to unify different fuzzy interpolation methods. The general structure of the FRI toolbox presented in **Fig. 30** , following this structure, the FRI toolbox could be run and evaluated the current FRI methods.

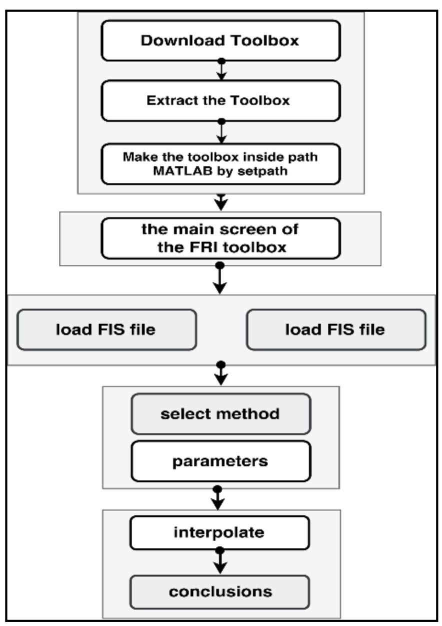

Fig. 30. *The General Structure of FRI Toolbox*

The current version of the FRI toolbox is available to download in [26]; it includes the following methods (KH, KH Stabilized, MACI, IMUL, CRF, VKK, GM, FRIPOC, LESFRI, and SCALE MOVE). The package of FRI toolbox contains software with a graphical user interface providing easy-to-use access, as shown in **Fig. 31**.

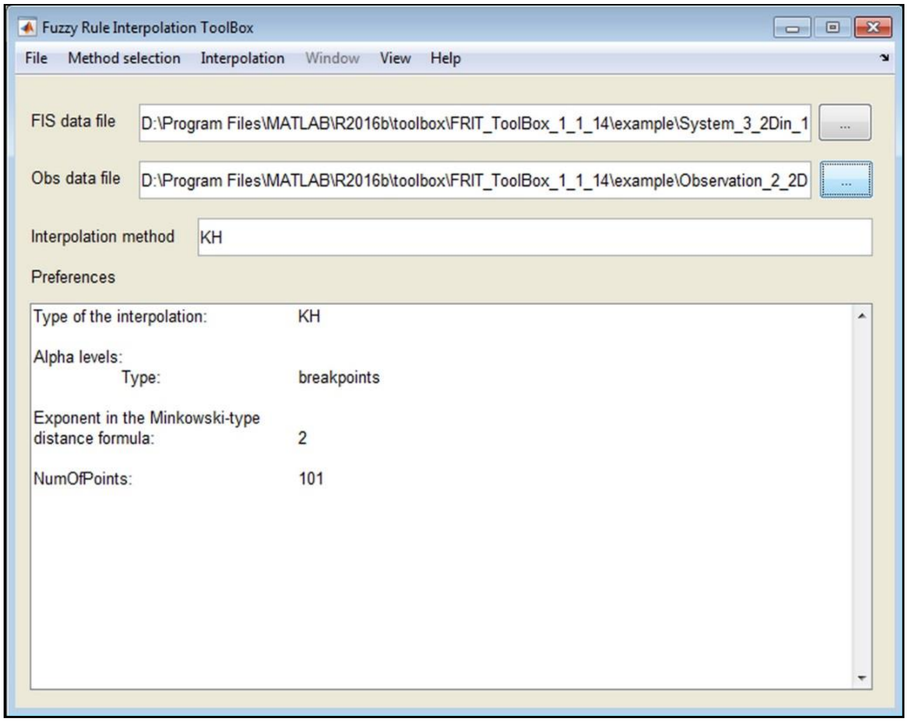

Fig. 31. *The Main Panel of the FRI MATLAB Toolbox [24]*

In the FRI toolbox, the Fuzzy Inference System (FIS) and OBServation (OBS) structure were different from the classical inference system. **Fig. 32** presents an example of FIS within the FRI toolbox. It worths mentioning that the fuzzy sets have to be convex and normal [3], [25].

```
MF1='A_{1;1}':'trimf',[10  20  30]![0 1 0]
MF2='A_{2;1}':'trapmf',[4.5  5  5.5  6]![0 1 1 0]
MF3='B_{1;1}':'singlmf',[0.46]![1]
```

Fig. 32. The New Parameters of the Membership Functions That Are Used by the File System in FRI Toolbox

Where the (trimf), (trapmf), and (singlmf) denote the triangular, trapezoidal, and singleton shapes of the fuzzy sets, respectively, the $A_{1;1}$, $A_{2;1}$ *and* $B_{1;1}$ refer to labels of the fuzzy sets of antecedents and consequent parts. The values [10 20 30], [4.5 5 5.5 6] and [0.46] denote the characteristic points (params) of the fuzzy sets in the universe of the discourse, where the triangular shape takes three values $[a_0, a_1, a_2]$, the trapezoidal shape represents by four values $[a_0, a_1, a_2, a_3]$, and singleton shape describes by one value $[a_0]$.

The new parameter in FIS general structure is called (paramsy), the characteristic points of the fuzzy sets in case of piecewise linear membership functions as (triangular, trapezoidal, and singleton) could be represented based on α-cut levels. The lower level will take the value *0* and the upper level will take the value *1*. For example, the new parameter of the trapezoidal shape can be represented based on the characteristic points [a_0, a_1, a_2, a_3], where the points [a_0, a_3] refer to the level *0* (lower level) and the points [a_1 *and* a_2] refer to the level 1 (upper level). **Fig. 32** describes the new parameter for the trapezoidal membership function (trapmf) represented by [0 1 1 0].

3.4.1.1. Fuzzy rule interpolation Matlab toolbox and related work

In the following, we present some relevant works related to the use of the implementation of the FRI toolbox.

In [31], the fundamental concepts of a Fuzzy Rule Interpolation-based (FRI) RL method called FRIQ-learning were discussed with benchmarks. The interpolation within the knowledge-based allows the removal of less important, unnecessary information, while still keeping the system functional. A Fuzzy Rule Interpolation-based (FRI) RL method called FRIQ-learning is a method that possesses this feature. FRIQ learning is also suitable for knowledge extraction. The FIVE FRI method was used by handling the antecedent and consequent fuzzy partitions of the fuzzy rule-base as scaling functions (weighting factors), which turns the fuzzy interpolation to scaled crisp interpolation. The implementation of the FIVE FRI also appears in the FRI Toolbox [26].

The authors in [32] give a brief introduction to the FRI methods and description of the refreshed and extended version of the original fuzzy rule interpolation MATLAB Toolbox. The methods used in the FRI toolbox (KH, KH Stabilized, MACI, IMUL, CRF, VKK, GM, FRIPOC, LESFRI, and SCALEMOVE) were tested to compare them according to the abnormality and linearity criteria, based on different numerical benchmark examples.

The authors in [33] introduced the benefits of the FRI in the Intrusion Detection Systems (IDS) application area, for the design and implementation of the detection mechanism for Distributed Denial of Service (DDOS) attacks. The performance of the FRI-IDS application was compared to other common classification algorithms (support vector machine, neural network, random forest, and decision tree) used for detecting DDOS attacks on the same open-source test-bed environment. According to the results, the overall detection rate of the FRI-IDS is in pair with other methods. Consequently, the FRI inference system could be a suitable approach to be implemented as a detection mechanism for IDS; as it effectively decreases the false positive rate value.

In [34], authors introduced a detection approach for defining abnormality using the Fuzzy Rule Interpolation (FRI) methods with Simple Network Management Protocol (SNMP) Management Information Base (MIB) parameters. The implemented experiments were performed using Matlab and the Fuzzy Rule Interpolation Toolbox (FRIT) [24]. The FIVE method was chosen as the inference reasoning of the proposed detection approach.

Authors in [35], analyzed equations and notations related to piecewise linearity property (PWL), which is aimed to highlight the problematic properties of the KH-FRI method to prove its efficiency with piecewise linearity condition. The study presented the benchmark examples to be served as a baseline for testing other FRI methods against situations in which the linearity condition for KH-FRI is not fulfilled. In the study, the FRI Toolbox was used to test the benchmark examples for different FRI methods.

3.4.2. Fuzzy Rule Interpolation OCTAVE Toolbox

Fuzzy Rule Interpolation (FRI) systems have been successfully implemented for many real-world applications. The FRI is also able to work with sparse rule bases that may not cover specific observations. For this reason, the current FRI MATLAB toolbox is implementing the fuzzy inference systems "type-1". Currently, there is a deficiency of the FRI toolbox available for creating other types of fuzzy systems and other programming languages.

The FRI Toolbox is a set of FRI functions, which run under the MATLAB environment. It is easy to use and beneficial tool for demonstration and research purpose. However, the FRI MATLAB is restricted in terms of its availability (MATLAB needs to license fees) and its user base. These constraints, limit the appropriation practicable of the current MATLAB FRI toolbox. For this purpose, a new developer framework based on OCTAVE environment [29], [30], which includes FRI methods.

The FRI toolbox based on openly available OCTAVE language will be presented. As regarded, OCTAVE language gives the advantage of being openly available, and it is accessible to user from a broad variety of backgrounds. Moreover, the high-level mathematical languages such as OCTAVE and MATLAB, they provide built-in primitives for representing and manipulating vectors and matrices, which can be used to represent and manipulate fuzzy sets directly. Meanwhile, these languages provide large built-in graphical abilities for *2D* and *3D* plotting, which can be utilized to represent fuzzy sets.

3.4.2.1. General description

The OCTAVE toolbox is aimed to generally give at least the same set of features as it exists in the MATLAB FRI Toolbox being an open-source toolbox, it is hoped that in the overall feature richness and functionality of the toolbox will be increased in time through the open-access contribution.

The OCTFRI toolbox is the initial version of FRI techniques based on OCTAVE language, it includes FRI functions to evaluate FISs and OBSs files from the command line and OCTAVE scripts, it is executed by read FISs and OBS files and produce a graphical output of both the membership functions and the FIS output. The current version supports twelve FRI methods

(KH, KHstabilized, MACI, IMUL, CRF, FIVE, VKK, GM, FRIPOC, LESFRI, VEIN, and ScaleMove (note that ScaleMove style inference is currently supported only in the OCTFRI Toolbox)). Nevertheless, the number of included techniques is still growing.

The toolbox is available for download under General Public License (GNU) from the website [26], [27], [28]. As part of describing the FRI inference systems functionality, a wide variety of forms of membership functions are supported similar to those currently provided by MATLAB, such as singleton (singlmf), triangle (trimf), trapezoid (trapmf), polygon (polymf). Currently, the number of rules is not restricted, there are some restrictions to use FRI toolbox, only convex and normal fuzzy sets are allowed. **Fig. 33** describes the general structure of the OCTFRI toolbox that could be used to run the OCTFRI toolbox and evaluate the current FRI methods.

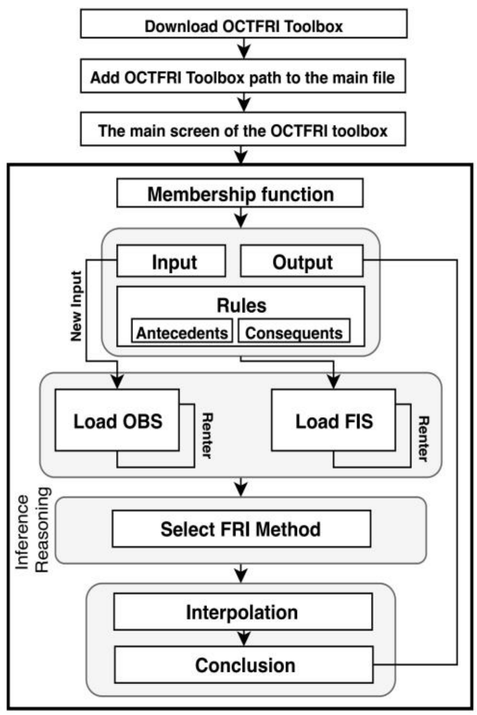

Fig. 33. *The General Structure of the OCTAVE FRI Toolbox*

3.4.2.2. Parameters of the FRI methods by OCTFRI toolbox

The FRI methods parameters could be defined by the main file of the OCTFRI toolbox "OCTFRI.m". Regarding the KH, VKK and KHstabilized methods which use two types of parameters, the first parameter is "breakpoints" (0 or 1), which is considered as a default parameter and denoted the α levels defined by the breakpoints [0, 1], the second one is "userdefined", which the user specifies the number of α levels that will be distributed uniformly in the interval [0,1], for example:

params.InterpolationType='KH';
params.AlphaLevels.Type='breakpoints';
or params.AlphaLevels.Type='userdefined';

In case MACI and IMUL methods that used RPtype (corecentre) parameter, which refer to the type of the reference point of the fuzzy set, e.g., params.InterpolationType='MACI'; params.RPtype='corecentre'. For FRIPOC and LESFRI methods share the same parameters, in which the FEAT-p technique can take all fuzzy sets that belong to the partition with different weight values. The type of the weighting factor and its parameters can also be set by the user, e.g.,

params.InterpolationType='FRIPOC';
params.NumOfPoints=501;
params.RPtype='corecentre';
params.NumOfpCuts=61;
params.SetInterpolationWeight.p=2;
params.ConsequentPositionWeight.p=2;

Most of FRI methods calculate multidimensional distances in the Minkowski sense. The parameter w of the formula can also be set by the user, where the default value is 2. The default values of the FRI methods parameters are sufficient to show the desired conclusion.

3.4.2.3. Usage of the OCTFRI toolbox and loading a FIS and OBS

The package of OCTFRI toolbox can be used from a graphical interface or from the command line. The current version of the OCTFRI is simple and easy to use, which contains all buttons of FRI methods, loading data files, and interpolation testing, as shown in **Fig. 34**.

OCTFRI toolbox can be begun by typing "OCTFRI.m" in the command line. The location of FIS and OBS data should be provided, which can be executed by the standard file open dialogue box as shown in **Fig. 35**.

3.4.2.4. Evaluation of FIS and OBS data files

The OCTFRI toolbox included twelve functions of the FRI methods, as presented in **Fig. 34** and six different examples of *"FIS"* and *"OBS"* files. In the following, two of FIS and OBS files selected to evaluate only eight of FRI methods. The inference process starts by loading the FIS and OBS data and select one of the FRI methods; then, the conclusion B^* could be shown by pressing on the interpolation button (see **Fig. 34**). The input and output universes will be shown in two separate windows, including the same number of diagrams as the dimension of the input and output.

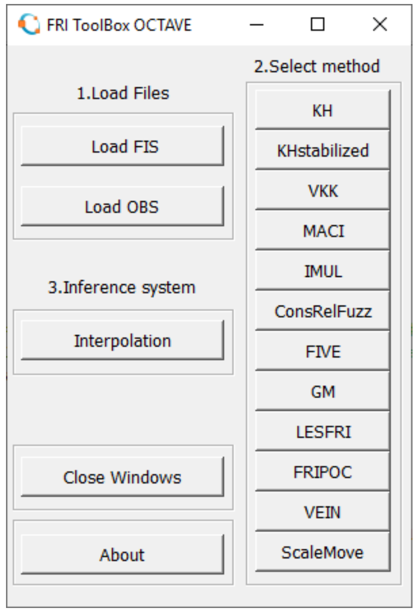

Fig. 34. *The Main Screen of the OCTFRI-TB*

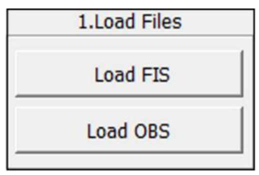

Fig. 35. *Location of FIS and OBS Data files*

- **Example "OCT1": Evaluation of the KH, KHSTAB, VKK and the ScaleMove FRI methods**

Example "OCT1" applies FIS1 and OBS1 data files stored in an OCTFRI folder including one input dimension, one output dimension, and only two fuzzy rules, as shown in **Fig. 36**. The membership functions of the input and output universes are triangular fuzzy sets. Example "OCT1" is tested on the KH, the KH Stabilized, the VKK, and the ScaleMove) methods, as described in **Fig. 37** and **Fig. 38**.

- **Example "OCT2": Evaluation the MACI, CRF, IMUL and GM FRI methods**

Example "OCT2" applies the FIS2 and OBS2 data files stored in OCTFRI folder including two input dimensions, one output dimension, and only four fuzzy rules, as shown in **Fig. 39**. The membership functions of the input and output universes are triangular fuzzy sets. Example "OCT2" is tested on the MACI, the CRF, the IMUL, and the GM) methods, as described in **Fig. 40** and **Fig. 41**.

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```
[System] %FIS FILE%
Name='FIS1'
Type='sparse'
Version=2.0
NumInputs=1
NumOutputs=1
NumRules=2
AndMethod=''
OrMethod=''
ImpMethod=''
AggMethod=''
DefuzzMethod='COG'

[Input1]
Name='input1'
Range=[0 50]
NumMFs=2
MF1='A1':'trimf',[5 10 15]![0 1 0]
MF2='A2':'trimf',[37 42 47]![0 1 0]

[Output1]
Name='output1'
Range=[0 50]
NumMFs=2
MF1='B1':'trimf',[5 10 15]![0 1 0]
MF2='B2':'trimf',[37 42 47]![0 1 0]

[Rules]
1, 1 (1) : 1
2, 2 (1) : 1

*************************
%OBS FILE%
NumInputs=1
ObsName='OBS1'
[Observation]
OBS1='A*_1':'trimf',[17 27 37]![0 1 0]
```

Fig. 36. *A FIS1 and OBS1 Stored in a File*

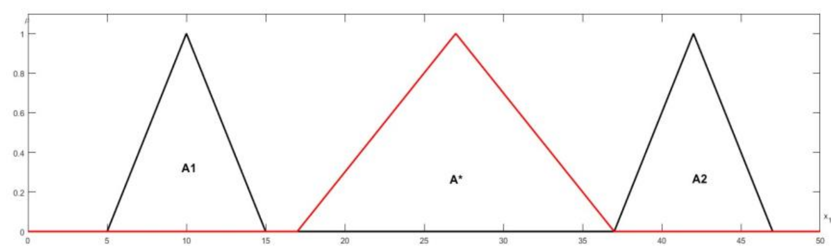

Fig. 37. *Antecedent Partitions and Observations for the Example "OCT1"*

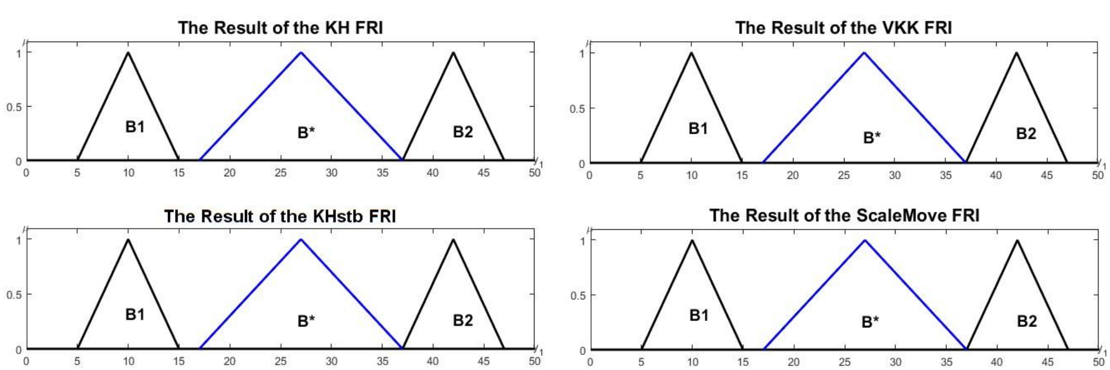

Fig. 38. *The Consequent Partitions and Approximate Conclusions of The FRI Methods for the Example "OCT1"*

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```
[System]  %FIS FILE%
Name='FIS2'
Type='sparse'
Version=2.0
NumInputs=2
NumOutputs=1
NumRules=4
AndMethod=''
OrMethod=''
ImpMethod=''
AggMethod=''
DefuzzMethod='COG'

[Input1]
Name='input1'
Range=[0 80]
NumMFs=4
MF1='mf1':'trapmf',[3 8 12 17]![0 1 1 0]
MF2='mf2':'trapmf',[23 28 32 37]![0 1 1 0]
MF3='mf3':'trapmf',[43 48 53 58]![0 1 1 0]
MF4='mf4':'trapmf',[63 68 72 77]![0 1 1 0]

[Input2]
Name='input2'
Range=[0 80]
NumMFs=4
MF1='A1':'trapmf',[3 8 12 17]![0 1 1 0]
MF2='A2':'trapmf',[23 28 32 37]![0 1 1 0]
MF3='A3':'trapmf',[43 48 53 58]![0 1 1 0]
MF4='A4':'trapmf',[63 68 72 77]![0 1 1 0]

[Output1]
Name='output1'
Range=[0 80]
NumMFs=4
MF1='B1':'trapmf',[3 8 12 17]![0 1 1 0]
MF2='B2':'trapmf',[23 28 32 37]![0 1 1 0]
MF3='B3':'trapmf',[43 48 53 58]![0 1 1 0]
MF4='B4':'trapmf',[63 68 72 77]![0 1 1 0]

[Rules]
1 1, 1 (1) : 1
2 2, 2 (1) : 1
3 3, 3 (1) : 1
4 4, 4 (1) : 1

*************************
%OBS FILE%
NumInputs=2
ObsName='OBS2'
[Observation]
OBS1='A*_1':'trapmf',[18 20 21 23]![0 1 1 0]
OBS2='A*_2':'trapmf',[37 39 40 42]![0 1 1 0]
```

Fig. 39. *A FIS2 and OBS2 Stored in a File*

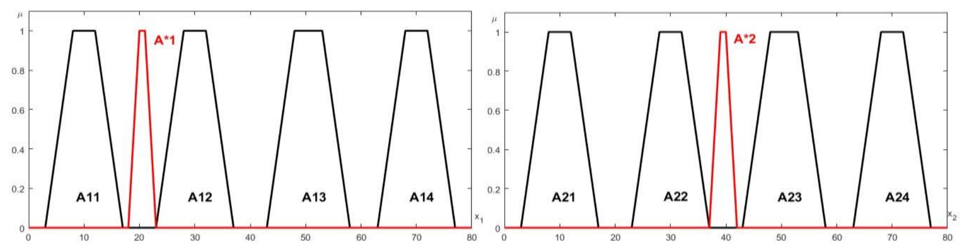

Fig. 40. *Antecedent Partitions and Observations for the Example "OCT2"*

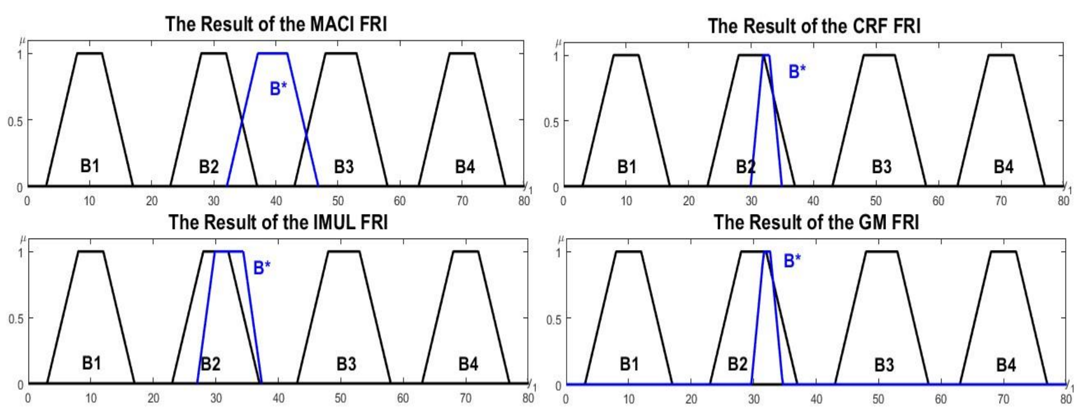

Fig. 41. *The Consequent Partitions and Approximate Conclusions of the FRI Methods for the Example "OCT2"*

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SUMMARY

This chapter provided some basic definitions of the complete and sparse fuzzy rule-base. The majority of the current FRI approaches were classified into two groups: one-step FRI and two-step FRI. Each group has been presented with a representative approach, as well as its extensions and improvements in detail, therefore, the chapter presented a survey of twelve different FRI approaches (KH, KH Stabilized, MACI, IMUL, CRF, VKK, GM, FRIPOC, LESFRI, and SCALEMOVE). The chapter also presented the refreshed and extended version of the original FRI MATLAB, which has been utilized effectively for the task of representing and handling fuzzy rule interpolation mechanisms. Finally, this chapter presented the initial version of the FRI toolbox based on OCTAVE language, which is open-source software (free license) available under the (GPL), it is compatible with MATLAB through packages and syntax.

The results introduced in this chapter are published in [32], [98].

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4. INTERPOLATIVE FUZZY REASONING METHOD BASED ON THE INCIRCLE OF A GENERALIZED TRIANGULAR FUZZY NUMBER

In this chapter, we will present a new fuzzy interpolation technique, which is called "Incircle-FRI", it can represent the reference point and the fuzziness sides of a triangular fuzzy number by its Incircle (Inscribed circle) properties. The proposed Incircle-FRI follows geometrical considerations for performing fuzzy interpolation, it takes care of producing Convex and Normal Fuzzy set (CNF) for all rules and observation configurations by presenting the main equations that prove the CNF property of the Incircle-FRI for all fuzzy rules and observation. To demonstrate the performance of the suggested FRI method, we present some numerical examples to compare the results of the Incircle-FRI with existing FRI methods (KH-FRI [3], [25], [39], KHstabilized-FRI [5], VKK-FRI [4], HCL-FRI [42], HTY-FRI [53], CCL-FRI [52], and to HS-FRI [22]) that will be discussed briefly in the chapter.

4.1. Background of fuzzy rule interpolative techniques and fuzzy numbers

In this subsection, we will present some basic concepts related to the fuzzy numbers and the suggested Incircle concept of the triangular fuzzy number.

4.1.1. Fuzzy numbers

A fuzzy set *A* defined on a universe of discourse *X*, which holds total ordering, is a fuzzy number, i.e., a CNF set; if it is normal, its height equals to one and convex. It has a membership grade of any elements between two other elements greater than or equal to the minimum membership degree of these two boundary elements. A convex fuzzy set can be defined by $\forall x$, $y \in U$, $\forall_\lambda \in [0, 1]$: $(\mu_A(\lambda.x + (1-\lambda.y)) \geq min\ (\mu_A(x), \mu_A(y)))$. The support of a fuzzy set *A* is the set of all elements in the universe of discourse *X* with a membership degree $\mu_A(x)$ is greater than zero. $(Supp(A)$: $x \in U$, $\mu_A(x) > 0)$.

The height of a fuzzy set is the maximum membership degree of all the elements of the universe, and it can be defined by $(Height(A)$: $max(x) \in U(\mu_A(x)))$. A fuzzy set is said to be normal if at least one element of the universe has a membership degree equal to *1*, $(\exists x \in U$, $\mu_A(x)$: $Height(A) = 1)$. The α-cut and the strong α-cut of a fuzzy set is the crisp subset of the universe in which the membership degrees are greater (strong α-cut), or greater, or equal (α-cut) than a specified α value. The α-cut can be represented by $(A_\alpha$: $x \in U$, $\mu_A(x) \geq \alpha)$, $\alpha \in [0, 1]$ and $(A_\alpha$: $x \in U$, $\mu_A(x) > \alpha)$, $\alpha \in [0, 1]$. The kernel of a fuzzy set is the crisp subset of the universe where the membership degrees are equal to 1. $(Kernel(A)$: $x \in U$, $\mu_A(x) = 1)$.

In case of a convex fuzzy set *A* on R_n, all of its α–cuts A_α are convex sets for all $\alpha \in (0, 1]$, i.e., its α–cuts are intervals.

In most cases, particular types of fuzzy numbers, such as trapezoidal and triangular could be used for real-life applications. A fuzzy number *A* is called a generalized trapezoidal fuzzy number if its membership function is given as follows:

$$\mu_A(X) = \begin{cases} \dfrac{H(X-a_1)}{(a_2-a_1)} \quad , \quad a_1 \le X \le a_2 \\ H \quad , \quad a_2 \le X \le a_3 \\ \dfrac{H(X-a_4)}{(a_3-a_4)} \quad , \quad a_3 \le X \le a_4 \\ 0 \quad , \quad X < a_1 \quad or \quad X > a_4 \end{cases} \tag{4.1}$$

where, $0 < H \le 1$. The generalized trapezoidal fuzzy number *A* is denoted by $A = (a_1, a_2, a_3, a_4; H)$ and has the shape of a trapezoid. A triangular fuzzy number is a particular case of generalized trapezoidal fuzzy number, having triangle-shaped membership function. Precisely, $A = (a_1, a_2, a_3, a_4; H)$ called a triangular fuzzy number if $a_2 = a_3$ as shown in **Fig. 42**.

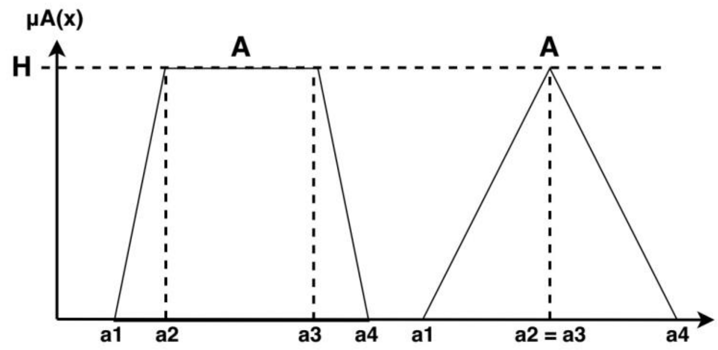

Fig. 42. *Generalized Trapezoidal and Triangular Fuzzy Numbers*

4.1.2. The Incircle of a Triangular fuzzy number

The Incircle of a triangular fuzzy number can be considered as an Incircle of a triangle. The Incircle of a triangle is that circle which touches all three main sides *(AB, BC, and AC)* of the triangle, and the points of tangency of the Incircle of ΔABC *(i.e., TA, TB, TC)* with its sides, as shown in **Fig. 43**.

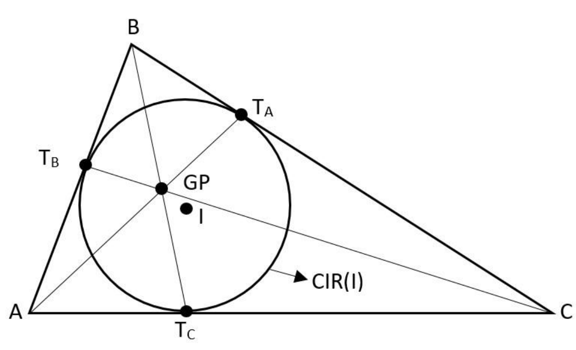

Fig. 43. *The Incircle CIR(I) with Gergonne Point GP in a Triangle ABC*

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By following the Ceva`s theorem directly (see theorem 1), we can define all properties of the incircle triangular fuzzy number, where cevians of particular importance in the general triangle (*medians, angle bisectors, Etc*.) are synchronous. Moreover, the fact that two tangents to a circle from a point outside the circle are equal.

Furthermore, the common point of a triangle *ABC* called a particular point of the triangle, it could be defined as the point of intersection of the cevians *ATA*, *BTB*, and *CTC*. Therefore, this point *"P"* will be later called the Gergonne Point (*GP*) of the triangle [54] as a reference point, which is the concurrence point for the cevians from the vertices to the points of tangency on the opposite sides of the triangle, as shown in **Fig. 44**.

Theorem 1. *The three cevians joining the vertices of a triangle to the point of tangency of the opposite sides with the Incircle are concurrent [55].*

Using Ceva's theorem, we directly get the following result: the given the existence of a triangle with vertices *A*, *B*, and *C*. The Trilinear Coordinates (*TCs*) of point *P*, which related to triangle *ABC* are three ordered numbers. Each corresponding to the distance from *P* to one of the sideline. *TCs* are generally referred via $(\alpha:\beta:\gamma)$, as shown in **Fig. 44**.

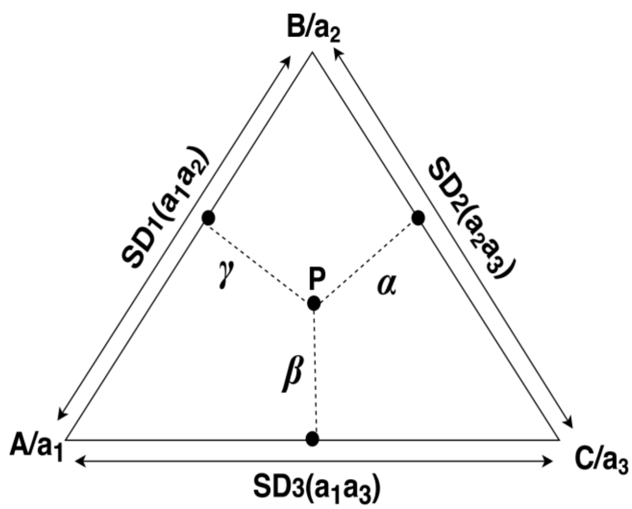

Fig. 44. *Trilinear Coordinates (TCs) αβγ of Point P*

If point *P* has *TCs* (α:β:γ), then the Cartesian Coordinates of *P* are calculated by **Eq.(4.2)**:

$$p = \left(\frac{1}{\alpha.\left|SD_{2(a2a3)}\right| + \beta.\left|SD_{3(a1a3)}\right| + \gamma.\left|SD_{1(a1a2)}\right|}\right).(\alpha.\left|SD_{2(a2a3)}\right|.A + \beta.\left|SD_{3(a1a3)}\right|.B + \gamma.\left|SD_{1(a1a2)}\right|.C) \quad (4.2)$$

TCs of the GP is given by **Eq.(4.3)**:

$$\frac{|SD_{3(a1a3)}|.|SD_{1(a1a2)}|}{|SD_{3(a1a3)}|+|SD_{1(a1a2)}|-|SD_{2(a2a3)}|}:$$
$$\frac{|SD_{2(a2a3)}|.|SD_{1(a1a2)}|}{|SD_{2(a2a3)}|+|SD_{1(a1a2)}|-|SD_{3(a1a3)}|}: \quad (4.3)$$
$$\frac{|SD_{2(a2a3)}|.|SD_{3(a1a3)}|}{|SD_{2(a2a3)}|+|SD_{3(a1a3)}|-|SD_{1(a1a2)}|}$$

Based on the properties of the Incircle triangular fuzzy number, some notations for the fuzzy rules and the observation fuzzy sets could be determined. It will be used to perform the approximate conclusion. The "*center*" of the triangle could be denoted by the Gergonne point (*GP*), which later will be named by "*Reference Point*".

The main sides of the triangular indicated by SD_1, SD_2, *and* SD_3. The lengths of tangents of the triangle can be determined by PS_1, PS_2, *and* PS_3, as shown in **Fig. 45**. These sides PS_1, PS_2, *and* PS_3 will be referred to as "*fuzziness sides*".

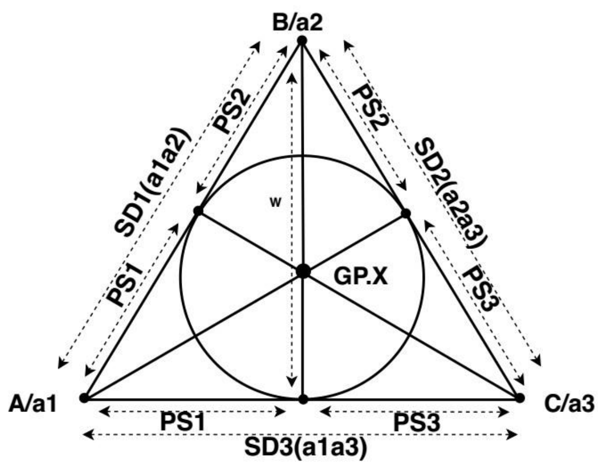

Fig. 45. *Triangular Fuzzy Number Notations*

Assuming that the triangular fuzzy set $A= (a_1, a_2, a_3; H)$, we get a triangle with the coordinates of vertices $A=(a_1,0)$, $B= (a_2, H)$ *and* $C= (a_3,0)$ with H. If H is equal to 1, then the fuzzy number is normal. The trapezoidal fuzzy set can be represented by two triangular fuzzy numbers $AL= (a_1, a_2, Mp; H)$ *and* $AR= (Mp, a_3, a_4; H)$, where Mp denotes the mid-point of the trapezoidal fuzzy set.

Some notations are required for calculating the approximating conclusion of the proposed Incircle-FRI is presented. Let us have a single-dimensional antecedent space and two adjacent

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fuzzy rules to the observation (like in the original KH-FRI) represented by triangular fuzzy sets as follows:

Incircle_Notation 1: The main sides of the triangular fuzzy set *A*, $SD_{1(a1a2)}$, $SD_{2(a2a3)}$, $SD_{3(a1a3)}$ can be calculated by **Eq.(4.4)**, where *H* denotes the height of the fuzzy set *A*.

$$\begin{aligned} |SD_1| &= \sqrt{(a_1 - a_2)^2 + H^2}, \\ |SD_2| &= \sqrt{(a_2 - a_3)^2 + H^2}, \\ |SD_3| &= a_3 - a_1 \end{aligned} \tag{4.4}$$

Incircle_Notation 2: Applying **Eq.(4.3)** and **Eq.(4.4)**, we can find the *TCs* and *GP* of a triangle fuzzy set *A*, which has vertices a_1, a_2, and a_3 as.

$$\begin{aligned} \alpha &= \frac{(a3 - a1).\sqrt{(a1 - a2)^2 + H^2}}{(a3 - a1) + \sqrt{(a1 - a2)^2 + H^2} - \sqrt{(a2 - a3)^2 + H^2}} \\ \beta &= \frac{\sqrt{(a2 - a3)^2 + H^2}.\sqrt{(a1 - a2)^2 + H^2}}{\sqrt{(a2 - a3)^2 + H^2} + \sqrt{(a1 - a2)^2 + H^2} - (a3 - a1)} \\ \gamma &= \frac{(a3 - a1).\sqrt{(a2 - a3)^2 + H^2}}{\sqrt{(a2 - a3)^2 + H^2} + (a3 - a1) - \sqrt{(a1 - a2)^2 + H^2}} \end{aligned} \tag{4.5}$$

Incircle_Notation 3: Using **Eq.(4.2)**, the Cartesian Coordinates (X_A, μ_A) of the reference point *GP* for triangle fuzzy set *A* can be calculated as follows.

$$GP_A = (X_A, \mu_A) = \left(\frac{a_1.\alpha|SD_2| + a_2.\beta|SD_3| + a_3.\gamma|SD_1|}{\alpha|SD_2| + \beta|SD_3| + \gamma|SD_1|}, \frac{\beta|SD_3|}{\alpha|SD_2| + \beta|SD_3| + \gamma|SD_1|} \right) \tag{4.6}$$

Regarding the height property of the fuzzy set, a fuzzy set *A* is normal if there is at least one element on the universe of discourse that has a membership degree equal to 1, ($\exists x \in U$, $\mu_A(x)$: $Height(A) = 1$).

Incircle_Notation 4: The left and right fuzziness lengths of the antecedent, consequent, and observation fuzzy sets could be determined by the fuzziness sides PS_1, PS_2, *and* PS_3, which can be calculated as follows.

$$A_{(PS1)} = \frac{(SD_1 + SD_3 - SD_2)}{2}$$
$$A_{(PS2)} = \frac{(SD_1 + SD_2 - SD_3)}{2} \qquad (4.7)$$
$$A_{(PS3)} = \frac{(SD_2 + SD_3 - SD_1)}{2}$$

4.2. A new fuzzy interpolative reasoning method based on Incircle of Triangular fuzzy number

In this subsection, a new fuzzy interpolative reasoning method will be presented for sparse fuzzy rule-based systems, which is based on the Incircle of the triangular fuzzy set of the fuzzy rules and observation. The representative values of the approximating conclusion fuzzy set will be determined by the reference points (GP_x), left fuzziness (PS_1), and right fuzziness (PS_3). In the following, the proposed Incircle-FRI will be discussed in detail.

4.2.1. Single antecedent variable with Triangular fuzzy sets

The triangular membership function is a particular case of the fuzzy sets that vastly used in fuzzy rule-based systems because of their simplicity. A triangular membership function describes by vertices $(a_1, a_2, a_3; H)$, where a_1 refers to the left side of the support, a_2 denotes the reference point, and a_3 refers to the right side of the support, and H denotes the height of the fuzzy set.

Fig. 46 illustrates the suggested reference point of the fuzzy set indicated by ($GP_X.A$) of fuzzy set A, the main left, right and base sides of triangle ABC indicated by SD_1, SD_2, and SD_3, respectively, and the fuzziness sides, where PS_1 refers to the left fuzziness, and PS_3 refers to the right fuzziness.

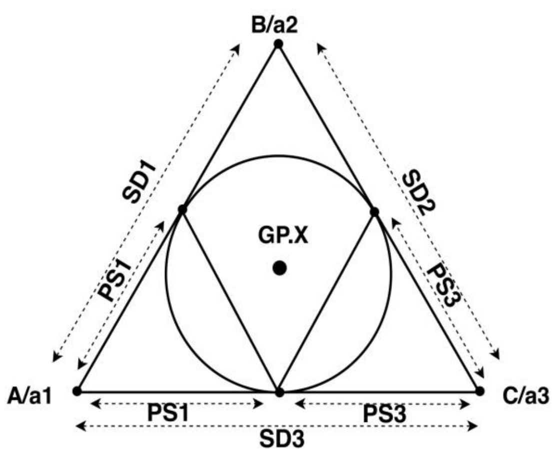

Fig. 46. *The Main Incircle_Notations of the Triangular Fuzzy Number Represented by GPx, SD_1, SD_2, SD_3, PS_1, PS_2 and PS_3*

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For simplicity, in the initial version of the suggested reasoning method, the two adjacent fuzzy rules $A_1 \Rightarrow B_1$ *and* $A_2 \Rightarrow B_2$ to the observation will be taken into consideration from the rule-base only. A_1, A_2, B_1, and B_2 denote the fuzzy sets of the antecedents and consequents, respectively. We assume that the observation fuzzy set A^* occurs between the fuzzy sets A_1 and A_2. The conclusion B^* fuzzy set denotes the fuzzy interpolative reasoning result, as shown in **Fig. 47**. The standard scheme of the fuzzy interpolative reasoning of the two fuzzy-rules and observation using triangular fuzzy sets can be defined as follows:

$$\begin{array}{c} \textit{Rule1: If } X \textit{ is } A_1 \textit{ then } Y \textit{ is } B_1 \\ \textit{Rule2: If } X \textit{ is } A_2 \textit{ then } Y \textit{ is } B_2 \\ \textit{Observation: } X \textit{ is } A^*. \\ \hline \textit{Conclusion: } Y = (B^*) \end{array}$$

where $Rule_1$ and $Rule_2$ are two adjacent fuzzy rules by $A_i = (a_{i,1}, a_{i,2}, a_{i,3})$, $B_i = (b_{i,1}, b_{i,2}, b_{i,3})$, where $(1 \le i \le 2)$, the observation $A^* = (a_1, a_2, a_3)$, and $B^* = (b_1, b_2, b_3)$.

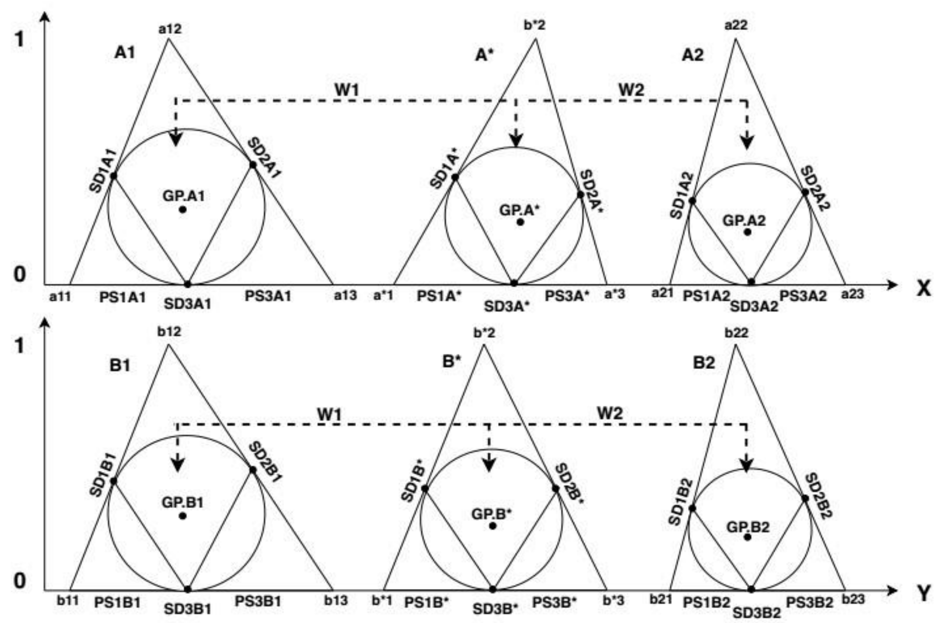

Fig. 47. *Fuzzy Interpolative Reasoning Using Triangular Membership Functions*

The fuzzy interpolative reasoning method "Incircle-FRI" related to single triangular fuzzy set could be determined, we need to compute the reference point and fuzziness sides by **Eq.(4.4)** - **Eq.(4.7)**, then applying the following steps:

Step 1: The two closest fuzzy rules [56] must be determined to perform interpolative for single antecedent fuzzy rules (m), $Rule_1, Rule_2, \ldots,$ and $Rule_m$, which are the nearest to the observation ($A_i \preccurlyeq A^* \preccurlyeq A_{i+1}$) as shown in **Fig. 47**. The closest fuzzy rules can be determined by the distances using **Eq.(4.8)**.

$$D = d(A_i, A^*) = d(GP_x.A_i, GP_x.A^*) \tag{4.8}$$

Where *d* refers to the distance between antecedents and observation fuzzy sets values, which is based on their reference points (*GPx*).

Step 2: Suppose the two adjacent fuzzy rules A_1 and A_2 are the left and right antecedent fuzzy sets to the observation fuzzy set A^*. Therefore, according to [56], the weight between the adjacent triangular fuzzy rules, $Rule_i$, and observation A^* can be determined as follows.

$$W_i = 1 - \frac{\left|GP_x.A^* - GP_x.A_i\right|}{GP_x.A_2 - GP_x.A_1} \tag{4.9}$$

GP_X refers to the reference point of the fuzzy sets; it could be calculated by **Eq.(4.6)**. W_i denotes the weight of $Rule_i$, which is used to perform interpolation between two surrounding rules to the observation using **Eq.(4.7)**. $(0 \le W_i \le 1),(1 \le i \le m)$, where $i = 1, 2$ represents the individual fuzzy rules as given in **Fig. 47** holding the property of ($W1 + W2 = 1$).

Step 3: The reference point (GP_X) of the fuzzy interpolative reasoning result B^* could be calculated based on the reference points of the consequents fuzzy sets and weights, shown by **Eq.(4.10)**.

$$GP_x.B^* = \sum_{i=1}^{2} W_i \times GP_x.B_i \tag{4.10}$$

The required reference points $GP_X.B_1$ and $GP_X.B_2$ can be calculated by **Eq.(4.6)**, *W1* and *W2* could be computed by **Eq.(4.9).**

Step 4: The fuzziness sides of the B^* fuzzy set could be calculated by **Eq.(4.11)**, which is based on the calculated fuzziness sides of the fuzzy adjacent rule bases and observation.

$$PS_M(B^*) = \begin{cases} PS_M(A^*) \times \sum_{i=1}^{2} W_i \times \frac{PS_M(B_i)}{PS_M(A_i)}, \\ \quad if \quad \exists_i \, PS_M(A_i) > 0 \\ PS_M(A^*), \\ \quad if \quad \forall_i PS_M(A_i) = 0 \end{cases} \tag{4.11}$$

Where $M \in [PS_1, PS_3]$, PS_1 refers to the left fuzziness side and PS_3 denotes the right fuzziness side of the triangular fuzzy set. If one of the antecedents fuzzy sets A_1 and A_2, the left PS_1 or right PS_3 are great than zero, the top part of **Eq.(4.11)** is implemented to conclude the left and right fuzziness sides PS_1 and PS_3 of the fuzzy set B^*. Otherwise, if both antecedents fuzzy sets A_1 and A_2 are found, the left PS_1 and right PS_3 are zero. I.e., in case of singleton fuzzy sets, the bottom part of **Eq.(4.11)** could be used.

Step 5: Now, based on the results of steps 3 and 4, the representative values of the conclusion B^* fuzzy set will be determined. The reference point is determined by **Eq.(4.10)**. The left fuzziness $[GP_X.B^* - B^*_{(PS1)}]$ and the right fuzziness side $[GP_X. B^* + B^*_{(PS3)}]$ are defined by **Eq.(4.11)**. Finally, the fuzzy interpolative reasoning conclusion B^* for the triangular fuzzy set can be collected as.

$$\begin{aligned} B_1^* &= GP_x.B^* - B^*_{(ps1)} \\ B_2^* &= GP_x.B^* \\ B_3^* &= GP_x.B^* + B^*_{(ps3)} \end{aligned} \tag{4.12}$$

Because $(B^*_1 \leq B^*_2 \leq B^*_3)$, we can see that the proposed method can preserve the convexity of the fuzzy interpolative reasoning result with the triangular fuzzy set. We can see that the value of the left fuzziness side is smaller than or equal to the value of the reference point (GP_X), which also is smaller than or equal to the value of the right fuzziness side.

In general, the result of the proposed "Incircle-FRI" is satisfied with logically consistent properties and concerning the ratios of fuzziness sides based on the two-fuzzy-rules interpolative reasoning technique, which obtained by **Eq.(4.11)**. The top equation of **Eq.(4.11)** is used to infer the fuzziness sides of the interpolated conclusion fuzzy set B^* if there exists a fuzzy rule whose fuzziness of the antecedent part is larger than zero. Otherwise, the bottom equation of **Eq.(4.11)** is used when the fuzziness sides of the antecedent part of the given fuzzy rules are zero. That means the larger the fuzziness of the membership function of a fuzzy set is the more fuzziness the fuzzy set has. The fuzzy interpolative reasoning (Incircle-FRI) result inferred by **Eq.(4.11)** satisfies the logically consistent properties for the ratios of fuzziness sides based on the two-fuzzy-rules interpolative reasoning technique, the ratio fuzziness sides

$RF.PS_m\ (A,\ B)$ of the consequence fuzzy sets B to antecedent fuzzy set A is computed in [52] as follows.

$$RF.PS_m(A,B) = \frac{RF.PS_m(B)}{RF.PS_m(A)} \tag{4.13}$$

Where $RF.PS_m(A) > 0$, $m \in$ the left $RF.PS_1$ *and* right $RF.PS_3$ fuzziness sides of the antecedent fuzzy set and the consequence fuzzy set of a fuzzy rule, respectively. $RF.PS_1(A, B)$ refers to the ratio fuzziness of the left fuzziness side B to the left fuzziness side A, $RF.PS_3(A, B)$ refers to the ratio fuzziness of the right fuzziness side B to the right fuzziness side A. For example; we can see that $RF.PS_1(A_1, B_1) = 1/2$, $RF.PS_1(A_2, B_2) = 2$, and $RF.PS_3(A_1, B_1) = RF.PS_3(A_2, B_2) = 1$.

The ratio of fuzziness sides $RF.PS_m(A, B)$ shown in **Eq.(4.13)**, it does not consider the situation that the antecedent fuzzy sets have vertical slopes at their left side or right side (i.e., $PS_1(A) = 0$ or $PS_3(A) = 0$). Thus, if there are two fuzzy rules $A_1 \Rightarrow B_1$, $A_2 \Rightarrow B_2$, and observation A^*, as shown in **Fig. 48**.

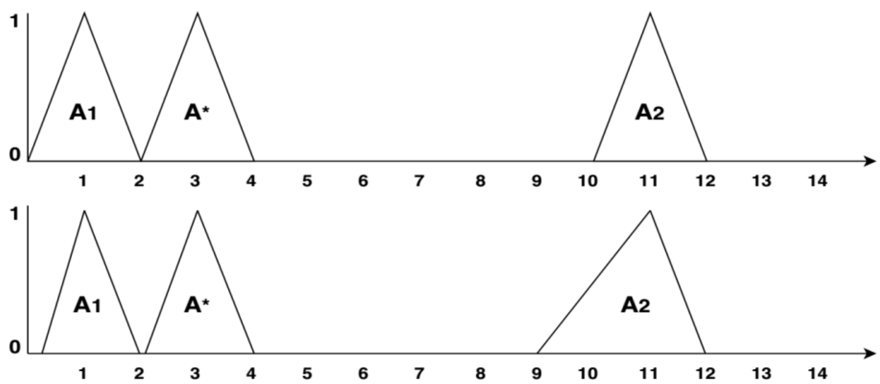

Fig. 48. *Fuzzy interpolative reasoning results for the gradual observations*

Suppose two neighboring fuzzy rules $A_1 \rightarrow B_1$, $A_2 \rightarrow B_2$, and one observation A^*, where A^* hits between A_1 and A_2. The fuzzy interpolative reasoning result B^* obtained by **Eq.(4.11)** satisfies the following two of the logically consistent properties.

Property 1: *Min* $(RF.PSm(A_1, B_1), RF.PSm(A_2, B_2)) \leq RF.PSm\ (A^*, B^*) \leq Max\ (RF.PSm(A_1, B_1), RF.PSm(A_2, B_2))$, *where* $m \in [ps1, ps3]$. *Then* $RF.PSm\ (A^*, B^*)$ *can be calculated by* ***Eq.(4.14)*** *as follows.*

$$RF.PS_m(A^*,B^*) = ((1-W2) \times RF.PS_m(A_1,B_1)) + (W2 \times RF.PS_m(A_2,B_2)) \tag{4.14}$$

Where the weight W_i between antecedents and observation could be defined by **Eq.(4.9)**.

Property 2: *If $RF.PSm(A_1, B_1) = RF.PSm(A_2, B_2) = K$, then $RF.PSm(A^*, B^*) = K$, where $K \geq 0$ and $m \in [PS_1, PS_3]$. Min $(RF.PSm(A_1, B_1), RF.PSm(A_2, B_2))$ = Max $(RF.PSm(A_1, B_1), RF.PSm(A_2, B_2)) = K$ and $RF.PSm(A^*, B^*) = K$.*

It is evident that, *Property 1* and *Property 2* are logically consistent concerning the left and right ratios of fuzziness sides *($RF.PS_1$ and $RF.PS_3$)* based on the two fuzzy rules. Based on **Eq.(4.11)**, the weight W_i of $RF.PSm(A_i, B_i)$ contributing to $RF.PSm(A^*, B^*)$ is determined by the distance of the reference points between A_i and A^*. The closer the reference point of A^* to the reference point of A_i, the larger weight of $RF.PSm(A_i, B_i)$, where $i = 1, 2$ and $m \in PS_1, PS_3$.

4.2.2. Single antecedent variable with Trapezoidal fuzzy sets

The Incircle concept of triangular fuzzy numbers can be extended to trapezoidal fuzzy set. A trapezoidal fuzzy set can be represented through two triangular fuzzy sets $AL = (a_1, a_2, Mp; H)$ and $AR = (Mp, a_3, a_4; H)$. Thus, the Incircle_Notations in **Eq.(4.4)**, **Eq.(4.6)**, and **Eq.(4.7)** will be used to calculate *AL (or TR_L)* and *AR (or TR_R)* separately.

Fig. 49 describes the left and right reference points of the trapezoidal fuzzy set that are denoted by $GP_X.AL$ and $GP_X.AR$. The main sides of the left triangle *AL* are described by SDL_1, SDL_2, and SDL_3, respectively. Furthermore, the fuzziness sides of *AL* can be described by the left PS_1AL and the right PS_3AL.

For the right triangle *AR*, the main sides are described by SDR_1, SDR_2, and SDR_3. Besides, the fuzziness sides of *AR* can be represented by the left PS_1AR and the right PS_3AR. Therefore, $GP_X.AL$, $GP_X.AR$, PS_1AL, PS_3AR, *and Mp* will be used to describe the left fuzziness and the right fuzziness of the trapezoidal membership function to determine the conclusion.

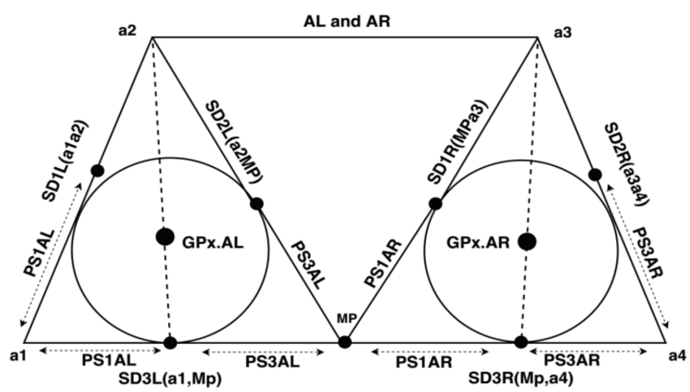

Fig. 49. *The Main Incircle_Notations of the Trapezoidal Fuzzy Number Represented by Two Triangular Fuzzy Sets AL and AR Notations.*

An example of the suggested fuzzy interpolative reasoning using trapezoidal fuzzy sets is shown in **Fig. 50**.

To interpolate the proposed Incircle fuzzy interpolative reasoning method with trapezoidal fuzzy sets (represented by two *AL* and *AR* triangular fuzzy sets), we need to find all

Incircle_Notations in **Eq.(4.4) - Eq.(4.7)** for each triangular (*AL*, *AR*), then applying the following steps:

Step 1: The two closest fuzzy rules [56] must be determined to perform interpolative for single antecedent fuzzy rules (m), $Rule_1$, $Rule_2$, ..., and $Rule_m$, which are nearest to the observation ($A_i \preccurlyeq A^* \preccurlyeq A_{i+1}$) as shown in **Fig. 50**. The closest fuzzy rules determined using the average of two reference points *GPx.AL* and *GPx.AR* of the trapezoidal fuzzy sets via ($AVG.GP_X=(GP_X.AL$ and $GP_X.AR)/2)$. Then, distances can be computed using **Eq.(4.15)**.

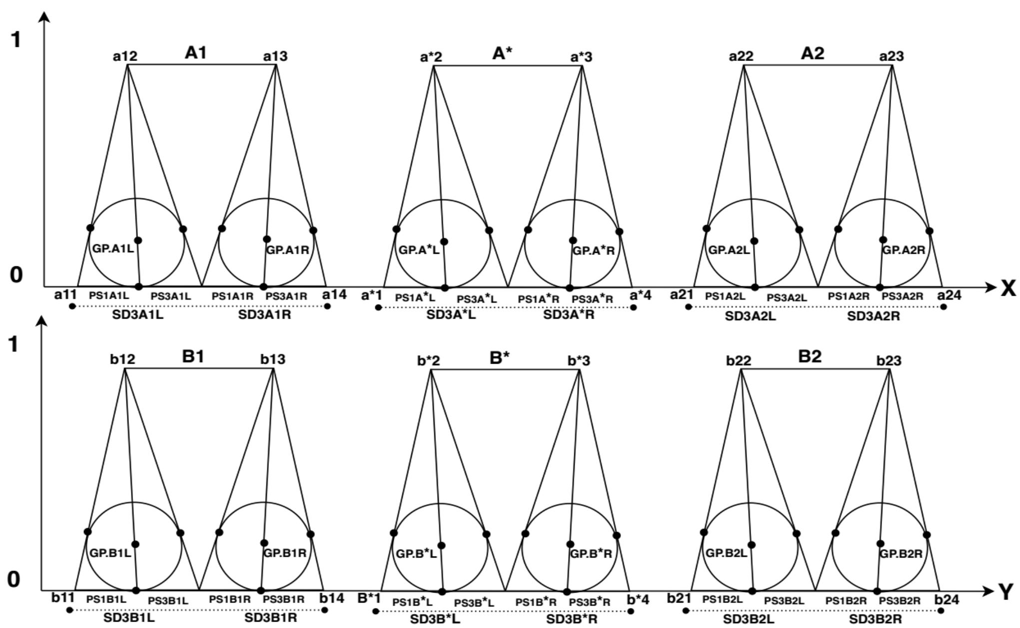

Fig. 50. *Fuzzy Interpolative Reasoning Using Trapezoidal Membership Functions*

$$D = d(A_i, A^*) = d(AVG.GP_x.A_i, AVG.GP_x.A^*) \tag{4.15}$$

Step 2: Supposing that two adjacent fuzzy rules A_1 and A_2 are the left and the right antecedent fuzzy sets to the observation fuzzy set A^*. Hence, the weight of trapezoidal fuzzy sets can be determined for each triangular (the left and right) by **Eq.(4.16)**.

$$WL_i = 1 - \frac{\left|GP_x.A^*L - GP_x.AL_i\right|}{GP_x.A_2L - GP_x.A_1L}, \qquad WR_i = 1 - \frac{\left|GP_x.A^*R - GP_x.AR_i\right|}{GP_x.A_2R - GP_x.A_1R} \tag{4.16}$$

WLi and *WRi* denote the weight of *Rulei* of the left and right triangulars fuzzy sets, ($0 \leq WLi \leq 1$), ($0 \leq WRi \leq 1$). Moreover, $i = 1, 2$, represents the individual fuzzy rules, as given in **Fig. 50** holding the property of ($WL1 + WL2 = 1$) *and* ($WR1 + WR2 = 1$).

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Step 3: The two reference points $GP_X.AL$ and $GP_X\ AR$ of the fuzzy interpolative reasoning result B^* could be calculated for left and right triangular by **Eq.(4.17)**.

$$GP_x.B^*L = \sum_{i=1}^{2} WL_i \times GP_x.BL_i, \qquad GP_x.B^*R = \sum_{i=1}^{2} WR_i \times GP_x.BR_i \tag{4.17}$$

Step 4: The fuzziness sides of the B^* fuzzy set can be calculated for the left triangular by **Eq.(4.18)** and for the right triangular by **Eq.(4.19)**.

$$PS_M(BL^*) = \begin{Bmatrix} PS_M(AL^*) \times \sum_{i=1}^{2} WL_i \times \frac{PS_M(BL_i)}{PS_M(AL_i)}, \\ \\ if\ \exists_i\ PS_M(AL_i) > 0 \\ PS_M(AL^*), \\ if\ \forall_i\ PS_M(AL_i) = 0 \end{Bmatrix} \tag{4.18}$$

$$PS_M(BR^*) = \begin{Bmatrix} PS_M(AR^*) \times \sum_{i=1}^{2} WR_i \times \frac{PS_M(BR_i)}{PS_M(AR_i)}, \\ \\ if\ \exists_i\ PS_M(AR_i) > 0 \\ PS_M(AR^*), \\ if\ \forall_i\ PS_M(AR_i) = 0 \end{Bmatrix} \tag{4.19}$$

$M \in [PS_1, PS_3]$, PS_1 refers to the left fuzziness side, and PS_3 denotes the right fuzziness side for the two triangles fuzzy sets AL and AR. The top part of **Eq.(4.18)** and **Eq.(4.19)** is performed to conclude the left fuzziness side PS_1 and the right fuzziness side PS_3 of the fuzziness of B^* fuzzy set. If one of the antecedents A_1 and A_2 fuzzy sets exits, PS_1 or PS_3 is bigger than zero, if both PS_1 or PS_3 of A_1 and A_2 fuzzy sets are zero, the bottom part of **Eq.(4.18)** and **Eq.(4.19)** could be used.

Step 5: Based on the results of step 3 and 4, the reference points ($GP_X.BL^*$ and $GP_X.BR^*$), the fuzziness sides (PS_1 and PS_3) could be used to determine the conclusion B^* of the trapezoidal fuzzy set, which based on the following cases

1. **In case** the results of the left and right B^*TR_L and $B^*\ TR_R$ triangulars fuzzy sets have the same values; then the conclusion can be determined as follows.

$$B^* = [GP_X.BL^*, GP_X.BL^*, GP_X.BR^*, GP_X.BR^*] \tag{4.20}$$

2. **In case** the result of the left triangular B^*TR_L has the same values, and the right triangular B^*TR_R has the same values too, but both left and right values are not the same. The conclusion values could be defined by $GP_{x1x2} = (GP_X.BL^* + GP_X.BR^*)/2$ as follows.

$$B^* = [GP_{x1x2},\ GP_{x1x2},\ GP_{x1x2},\ GP_{x1x2}] \quad (4.21)$$

3. **In case** the left has the same values, we will use B^*L, but in case the right has the same values, we will use B^*R, then the conclusion can be determined as follows.

$$\begin{aligned} BL^* &= [GP_{x.}B^*L,\ GP_{x.}B^*L,\ GP_{x.}B^*R,\ GP_{x.}B^*R+B^*R_{(ps3)}] \\ BR^* &= [GP_{x.}B^*L-B^*L_{(ps1)},\ GP_{x.}B^*L,\ GP_{x.}B^*R,\ GP_{x.}B^*R] \end{aligned} \quad (4.22)$$

4. **In case**, all values result from the left and the right triangles are different. The conclusion can be determined as follows.

$$\begin{aligned} B_1^* &= GP_x.B^*L - B^*L_{(PS1)} \\ B_2^* &= MP.B^* - B^*L_{(PS3)} \\ B_3^* &= MP.B^* - B^*R_{(PS1)} \\ B_4^* &= GP_x.B^*R + B^*R_{(PS3)} \end{aligned} \quad (4.23)$$

Because ($B^*_1 \leq B^*_2 \leq B^*_3 \leq B^*_4$), we can see that the proposed method can preserve the convexity of the fuzzy interpolative reasoning result with a trapezoidal fuzzy set. We can see that the value of the left point (B^*_1) is smaller than or equal to the values of the reference point ($GP_X.B^*L$ *and* $GP_X.B^*R$), which are also smaller than or equal to the value of the right point (B^*_4). Regarding the main point (MP) of the trapezoidal fuzzy set can be calculated by $MP.B^* = AVG.GP_{X.}B_1 + (((AVG.GP_{X.}A^* - AVG.GP_{X.}A_1) \times (AVG.GP_{X.}B_2 - AVG.GP_{X.}B1)) / (AVG.GP_{X.}A_2 - AVG.GP_{X.}A_1))$.

4.3. The validation of the Incircle-FRI to Normality and Convexity (CNF) condition

The proposed Incircle-FRI method developed related to the representation of the reference point and fuzziness sides of a triangular fuzzy number by its Incircle properties, this follows geometrical considerations for performing fuzzy interpolation, which leads to producing Convex and Normal Fuzzy set (CNF) for all the fuzzy rule bases and observation configurations. In order to ensure that ($Inf(A_{1\alpha}) \leq Inf(A_{2\alpha}) \leq Sup(A_{1\alpha}) \leq Sup(A_{2\alpha})$), the coordinate of the conclusion B^* should be satisfying with ($b^*_1 \leq b^*_2 \leq b^*_3 \leq b^*_4$).

In the following, we will study all CNF_Notations that used to prove the normality and convexity of the Incircle-FRI method according to the core and fuzziness sides conditions as follows:

If the fuzzy sets of the antecedent and the consequent have a uniform core and boundary lengths, then the conclusion fuzzy set is always normal if and only if the following conditions of the Left Fuzziness Side ($LFS.PS_1$) length of the observation by **Eq.(4.24)** and **Eq.(4.25)** hold.

- If $LFS(A^*) \neq 0$

$$LFS(PS_1)1 \leq LFS(PS_1)2 \quad (4.24)$$

where

$$LFS(PS_1)1 = LFS\_diff(B_1B_2) \times (LFS(A) - LFS(A^*)),$$
$$LFS(PS_1)2 = LFS(B) \times (LFS\_diff(A^*A_1) + LFS\_diff(A_2A^*) + 2 \times LFS(A^*))$$

- If $LFS\ (A^*) = 0$

$$LFS(PS_1)1 \leq LFS(PS_1)2 \quad (4.25)$$

where

$$LFS(PS_1)1 = LFS\_diff(B_1B_2) \times (LFS(A) - LFS(A^*)),$$
$$LFS(PS_1)2 = LFS(B) \times LFS\_diff(A_1A_2)$$

And,

$$LFS(PS_1)(A) = (LFS(PS_1)(A_1) + LFS(PS_1)(A_2))/2, \text{ and}$$
$$LFS(PS_1)(B) = (LFS(PS_1)(B_1) + LFS(PS_1)(B_2))/2.$$

For Core (*CoreL and CoreR*) and Right Fuzziness Side ($RFS(PS_3)$) lengths, similar equations to the left fuzziness side ($LFS(PS_1)$) length could be constructed.

According to **Eq.(4.26)**, **Eq.(4.27)**, and **Eq.(4.28)**, the core and fuzziness sides lengths (Left and Right) of the conclusion can be determined. For verifying the normality of the Left Fuzziness Side ($LFS(PS_1)$) length of the conclusion**,** **Eq.(4.26)** could be applied:

$$LFS(PS_1)1 \leq LFS(PS_1)2 \quad (4.26)$$

where

$$LFS(PS_1)1 = LFS\_diff(B_1B_2) \times (((LFS(A_1) + LFS\_diff(A^*A_1)) \times (LFS(A_2) + LFS\_diff(A_2A^*))) - ((LFS(A^*) + LFS\_diff(A^*A_1)) \times (LFS(A^*) + LFS\_diff(A_2A^*))))$$

$$LFS(PS_1)2 = ((LFS(A_1) + LFS\_diff(A^*A_1)) \times (LFS\_diff(A^*A_1) + LFS(A^*)) \times LFS(B_2)) + ((LFS(A_2) + LFS\_diff(A_2A^*)) \times (LFS\_diff(A_2A^*) + LFS(A^*)) \times LFS(B_1))$$

The Core length of the conclusion can be determined by **Eq.(4.27)** as follows:

For the left core $CoreL(GP)$ of TR_L.

$$CoreL(GP)1 \leq CoreL(GP)2 \quad (4.27)$$

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where

$$CoreL(GP)1 = CoreL\_diff(B_1B_2) \times (((CoreL(A_1) + CoreL\_diff(A^*A_1)) \times (CoreL(A_2) + CoreL\_diff(A_2A^*))) - ((CoreL(A^*) + CoreL\_diff(A^*A_1)) \times (CoreL(A^*) + CoreL\_diff(A_2A^*))))$$

$$CoreL(GP)2 = ((CoreL(A_1) + CoreL\_diff(A^*A_1)) \times (CoreL\_diff(A^*A_1) + CoreL(A^*)) \times CoreL(B_2)) + ((CoreL(A_2) + CoreL\_diff(A_2A^*)) \times (CoreL\_diff(A_2A^*) + CoreL(A^*)) \times CoreL(B_1))$$

For the right core length ("*CoreR(GP)*") can be calculated as the same equations and parameters of the left core TR_L "*CoreL(GP)*" as presented above.
For Right Fuzziness Side *(RFS(PS₃))* length of the conclusion can be determined by the following **Eq.(4.28)**.

$$RFS(PS_3)1 \leq RFS(PS_3)2 \qquad (4.28)$$

where

$$RFS(PS_3)1 = RFS\_diff(B_1B_2) \times (((RFS(A_1) + RFS\_diff(A^*A_1)) \times (RFS(A_2) + RFS\_diff(A_2A^*))) - ((RFS(A^*) + RFS\_diff(A^*A_1)) \times (RFS(A^*) + RFS\_diff(A_2A^*))))$$

$$RFS(PS_3)2 = ((RFS(A_1) + RFS\_diff(A^*A_1)) \times (RFS\_diff(A^*A_1) + RFS(A^*)) \times RFS(B_2)) + ((RFS(A_2) + RFS\_diff(A_2A^*)) \times (RFS\_diff(A_2A^*) + RFS(A^*)) \times RFS(B_1))$$

The parameters of the core, left, and right lengths of the previous equations can be defined as follows:

$$LFS = TR_L PS_1(F_s)$$
$$CoreL = TR_L GP_X (F_s)$$
$$CoreR = TR_R GP_X (F_s)$$
$$RFS = TR_R PS_3(F_s)$$

F_S belongs to fuzzy rules and observation $(A_i, A^*, \text{and } B_i)$, and TR_L and TR_R refer to the left and right triangular in case the membership function is trapezoidal.

$$LFS\_diff(A^*A_1) = TR_L PS_1(A^*) - TR_L PS_1(A_1)$$
$$CoreL\_diff(A^*A_1) = Core\_TR_L L(A^*) - Core\_TR_L L(A_1)$$
$$CoreR\_diff(A^*A_1) = Core\_TR_R R(A^*) - Core\_TR_R R(A_1)$$
$$RFS\_diff(A^*A_1) = TR_R PS_3(A^*) - TR_R PS_3(A_1)$$

In the same way, we can calculate the left, right and core between (A_2A^*, B_1B_2, A_2A_1), Moreover, from another point of view, the length ratio of the distance between the fuzzy sets of the antecedent with observation (A_i, A^*), and consequent (B_i). **Eq.(4.29)**, **Eq.(4.30)**, and **Eq.(4.31)** could also be used to check the normality (validity) of the conclusion, which can be defined as follows:

For the length ratio of the left fuzziness side ($LFS(PS_1)$):

$$\begin{aligned} &LFS.Ratio1 = LFS(B) / LFS(A) \\ &LFS.Ratio2 = LFS(A) / (LFS(A^*A_1) + LFS(A_2A^*)) \end{aligned} \tag{4.29}$$

where

$$\begin{aligned} &LFS\ (A) = TR_LPS_1(A_2) - TR_LPS_1(A_1) \\ &LFS(B) = TR_LPS_1(B_2) - TR_LPS_1(B_1) \\ &LFS(A^*A_1) = TR_LPS_1(A^*) - TR_LPS_1(A_1) \\ &LFS(A_2A^*) = TR_LPS_1(A_2) - TR_LPS_1(A^*) \end{aligned}$$

For the length ratio of the Left Core of TR_L ($CoreL.Ratio$):

$$\begin{aligned} &CoreL.Ratio1 = CoreL(B) / CoreL(A) \\ &CoreL.Ratio2 = CoreL(A) / (CoreL(A^*A_1) + CoreL(A_2A^*)) \end{aligned} \tag{4.30}$$

where

$$\begin{aligned} &CoreL(A) = CoreL(A_2) - CoreL(A_1) \\ &CoreL(B) = CoreL(B_2) - CoreL(B_1) \\ &CoreL(A^*A_1) = CoreL(A^*) - CoreL(A_1) \\ &CoreL(A_2A^*) = CoreL(A_2) - CoreL(A^*) \end{aligned}$$

The Right Core of TR_R ($CoreR.Ratio$), it can be calculated at the same ($CoreL.Ratio$).

For the length ratio of the right fuzziness side ($RFS(PS_3)$):

$$\begin{aligned} &RFS.Ratio1 = RFS(B) / RFS(A) \\ &RFS.Ratio2 = RFS(A) / (RFS(A^*A_1) + RFS(A_2A^*)) \end{aligned} \tag{4.31}$$

where

$$\begin{aligned} &RFS\ (A) = TR_RPS_3(A_2) - TR_RPS_3(A_1) \\ &RFS(B) = TR_RPS_3(B_2) - TR_RPS_3(B_1) \\ &RFS(A^*A_1) = TR_RPS_3(A^*) - TR_RPS_3(A_1) \\ &RFS(A_2A^*) = TR_RPS_3(A_2) - TR_RPS_3(A^*) \end{aligned}$$

4.4. Comparison of the proposed method with some other FRI methods

This subsection discusses the performance of the proposed "Incircle-FRI" method, some numerical examples in [22], [52], [57], [58], [59] will be compared with the results of KH-FRI

[3], [25], [39], KHstabilized-FRI [5], VKK-FRI [4], CCL-FRI [52], HS-FRI [22], HTY-FRI [53], and HCL-FRI [42]. The KH-FRI, the KHstabilized-FRI, and the VKK-FRI methods were tested using the Matlab FRI toolbox [24], [26]. Also, we present a comparison summary of the selected FRI methods with Incircle-FRI based on five evaluation criteria (i.e., "CNF property", "different membership functions", which mean there is no restriction the shape of the fuzzy sets, "different kinds membership functions of the antecedents and the consequents fuzzy rules", "the Approximation capability of the fuzziness and core between the observation and conclusion", and "logically consistent with respect to the ratios of fuzziness sides (see property 1 and property 2)").

Example TR1 [22], [52], [57], [58], [59]:

- All the rule antecedents, consequents, and the observation are triangular fuzzy sets.
- Observation $A^* = [7, 8, 9]$.

Table 1 describes all the attribute values and the results of fuzzy interpolative reasoning methods, therefore, we conclude the following:

- The conclusion fuzzy set of the proposed Incircle-FRI is represented by triangular membership function as $B^* = (GP_X.B^*\text{-}PS_1, GP_X.B^*, GP_X.B^*+PS_3)$.
- Based on **Eq.(4.6)**, **Eq.(4.9)**, and **Eq.(4.10),** the reference point value was calculated by the fuzzy interpolation of the fuzzy sets A_1, A_2, B_1, and B_2. Where $GP_X.B^* = 5.42$.
- Also, based on **Eq.(4.7)**, **Eq.(4.9)**, and **Eq.(4.11)**, we obtain the left fuzziness $PS_1 = 0.472$ and the right fuzziness $PS_3 = 1.737$.
- The proposed Incircle-FRI gets the interpolated consequence as a triangular fuzzy set $B^* = [4.95\ 5.42\ 7.16]$.
- **Fig. 100** (*see Appendix A.1*) describes the results of various FRI methods, in which, the KH [3], [25], [39], KHstabilized [5], and VKK [4] FRI methods generated an abnormal (non-convex) fuzzy sets. While the FRI methods CCL [52], HS [22], HTY [53], HCL [42], and the proposed Incircle-FRI have CNF conclusion.

Based on **Eq.(4.13)** and **Eq.(4.14)** and **Table 1**, the logically consistent properties and the respect to ratios of the fuzziness two adjacent rules ($A_1 \Rightarrow B_1$) and ($A_2 \Rightarrow B_2$), we conclude the following:

- The left ratio fuzziness $RF.PS_1\,(A_1, B_1) = 0.4$ and the ratio $RF.PS_1\,(A_2, B_2) = 0.5$, and the left ratio fuzziness $RF.PS_1\,(A^*, B^*)$ of the proposed Incircle-FRI, CCL FRI, the HCL FRI, the HTY-FRI and HS-FRI methods are 0.44, 0.44, 0.22, 0.66 and 0.43,
  Therefore, the proposed Incircle-FRI, CCL FRI, and HS-FRI satisfy property 1, where $MIN\ (RF.PS_1\ (A_1,\ B_1),\ RF.PS_1\ (A_2,\ B_2)) = 0.4 \leq RF.PS_1\ (A^*,\ B^*) = [Incircle(0.44), CCL(0.44), HS(0.43)] \leq MAX\ (RF.PS_1\,(A_1, B_1), RF.PS_1\,(A_2, B_2)) = 0.5$.
- The right ratio fuzziness of rules $(A_1 \Rightarrow B_1$ *and* $A_2 \Rightarrow B_2)$ $RF.PS_3\,(A_1, B_1) = RF.PS_3\,(A_2, B_2) = 2$, and the right ratio fuzziness $RF.PS_3\,(A^*, B^*)$ of the proposed Incircle-FRI, CCL FRI,

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the HCL FRI, the HTY FRI, HS FRI, KHstabilized-FRI, and KH-FRI methods are 2, 2, 0.8, 0.88, 1.12, 2, and 2, respectively.

We can see that only $RF.PS_3\ (A^*, B^*)$ of the proposed Incircle-FRI, CCL FRI, KHstabilized-FRI, and KH-FRI satisfy Property 2, where $MIN\ (RF.PS_3\ (A_1, B_1), RF.PS_3\ (A_2, B_2)) = MAX\ (RF.PS_3\ (A_1, B_1), RF.PS_3\ (A_2, B_2)) = RF.PS_3\ (A^*, B^*) = 2$.

Based on the above, the fuzzy interpolative reasoning result of the proposed Incircle-FRI is logically consistent in terms of Property 1 and property 2.

Table 1. *Fuzzy Interpolative Reasoning Results of Example TR1*

Attribute Values	**Methods**	**Results of Fuzzy Interpolative reasoning**
A_1=[0 5 6] A_2=[11 13 14] B_1=[0 2 4] B_2=[10 11 13] A^*=[7 8 9]	KH-FRI [3],[25],[39] KHstabilized-FRI [5] VKK-FRI [4] CCL-FRI [52] HS-FRI [22] HTY-FRI [53] HCL-FRI [42]	B^*=(6.36 5.38 7.38) B^*=(6.36 5.38 7.38) B^*=(6.15 5.38 7.84) B^*=(4.94 5.38 7.38) B^*=(5.83 6.26 7.38) B^*=(5.76 6.42 7.30) B^*=(6.36 6.58 7.38)
The Incircle-FRI		**B^*=(4.95 5.42 7.16)**

Example TR2 [22], [52], [58]:

- All the rule antecedents, consequents, and the observation are triangular fuzzy sets.
- Observation $A^* = [8, 8, 8]$.

Table 2 and **Fig. 101** (see Appendix *A.2*

) give all attribute values and the results of fuzzy interpolative reasoning, we conclude the following:

- In which the KH-FRI [3], [25], [39], the KHstabilized-FRI [5], and the VKK-FRI [4], once again generate a non-convex fuzzy conclusion. The HCL-FRI [42] produces a non-convex fuzzy set. Although it is a non-triangular fuzzy set, the FRI methods CCL-FRI [52], HS-FRI [22], HTY-FRI [53], and proposed Incircle-FRI generate normal singleton conclusion.
- By **Eq.(4.10)**, the Incircle-FRI produces a singleton fuzzy set represented by the reference point $GPx.B^* = 5.42$.
- By **Eq.(4.11)**, the left fuzziness $PS_1=0$ and the right fuzziness $PS_3=0$.
- The proposed Incircle-FRI gets the interpolated consequence as singleton fuzzy set $B^* = [5.42, 5.42, 5.42]$ based on **Eq.(4.12)**.
- In this example, the proposed Incircle-FRI performs better than the KH, KHstabilized, VKK, and HCL FRI.

Based on **Eq.(4.13)** and **Eq.(4.14)** and **Table 2**, the left ratio fuzziness, and the right ratio fuzziness of this example have the same results in Example TR1 as follows:

- The results of the proposed Incircle-FRI, CCL-FRI satisfy property 1 for the left ratio fuzziness $RF.PS_1$ (A^*, B^*), where $MIN (RF.PS_1 (A_1, B_1), RF.PS_1 (A_2, B_2)) = 0.4 \leq RF.PS_1 (A^*, B^*) = [Incircle (0.44), CCL (0.44)] \leq MAX (RF.PS_1 (A_1, B_1), RF.PS_1 (A_2, B_2)) = 0.5$.
- While the right ratio fuzziness $RF.PS_3 (A_1, B_1) = RF.PS_3 (A_2, B_2) = 2$, and the results of the right ratio fuzziness $RF.PS_3 (A^*, B^*)$ of the proposed Incircle-FRI and CCL-FRI satisfy Property 2, which are 2.

Based on the above, the fuzzy interpolative reasoning result of the proposed Incircle-FRI is logically consistent in terms of Property 1 and Property 2 for the left and right ratio fuzziness, respectively.

Table 2. *Fuzzy Interpolative Reasoning Results of Example TR2*

Attribute Values	**Methods**	**Results of Fuzzy Interpolative reasoning**
A_1=[0 5 6] A_2=[11 13 14] B_1=[0 2 4] B_2=[10 11 13] A^*=[8 8 8]	KH-FRI [3],[25],[39] KHstabilized-FRI [5] VKK-FRI [4] CCL-FRI [52] HS-FRI [22] HTY-FRI [53] HCL-FRI [42]	B^*=(7.27 5.38 6.25) B^*=(7.27 5.38 6.25) B^*=(7.00 5.38 7.00) B^*=(5.38 5.38 5.38) B^*=(6.49 6.49 6.49) B^*=(6.49 6.49 6.49) B^*=(7.27 - 6.25)
The Incircle-FRI		**B^*=(5.42 5.42 5.42)**

Note: the sign (-) indicates no clear evidence for the method to handle the case in the example

Example TR3 [22], [52], [58]:

- In this example, the antecedents of the fuzzy rules and observation are represented by singleton and triangular fuzzy sets.
- Observation $A^* = [5, 6, 8]$.

Table 3 and **Fig. 102** (*see Appendix A.3*) describe all the results and attributes values of this example; therefore, we conclude the following:

- The result of the proposed Incircle-FRI is computed using **Eq.(4.6)**, **Eq.(4.9)**, and **Eq.(4.10)**, which produce a triangular fuzzy set, in which the value of the reference point $GP_X.B^*$ is calculated based on the fuzzy interpolation of the fuzzy sets A_1, A_2, B_1, and B_2, where $GP_X.B^* = 6.37$.
- According to **Eq.(4.7)**, **Eq.(4.9)**, and **Eq.(4.11)**, the left fuzziness $PS_1 = 1.089$, and the right fuzziness $PS_3 = 1.910$ are determined.
- Based on **Eq.(4.12)**, the conclusion of the proposed Incircle-FRI is a CNF triangular fuzzy set $B^* = [5.28, 6.37, 8.28]$.
- Meanwhile, the KH-FRI [3], [25], [39], the KHstabilized-FRI [5], the HCL-FRI [42], the CCL-FRI [52], the HS-FRI [22], and HTY-FRI [53] give a very reasonable conclusion. In contrast, in this case, the VKK-FRI [4] and the HTY-FRI [53] could not generate a usable conclusion.

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Example TP1 [22], [52], [58]:

- All the rule antecedents, consequents, and the observation are trapezoidal fuzzy sets.
- Observation $A^* = [6, 6, 9, 10]$.

Table 4 and **Fig. 103** (*see Appendix A.4*) describe all the attribute values and the results of FRI methods, therefore, we conclude the following:

- Based on **Eq.(4.6)**, **Eq.(4.16)**, and **Eq.(4.17),** the reference points values are calculated by the fuzzy interpolation of the fuzzy sets A_1, A_2, B_1, and B_2, where $GP_X.BL^* = 4.54$ and $GP_X.BR^* = 7.47$.

Table 3. *Fuzzy Interpolative Reasoning Results of Example TR3*

Attribute Values	**Methods**	**Results of Fuzzy Interpolative reasoning**
A_1=[3 3 3] A_2=[12 12 12] B_1=[4 4 4] B_2=[10 11 13] A^*=[5 6 8]	KH-FRI [3],[25],[39] KHstabilized-FRI [5] VKK-FRI [4] CCL-FRI [52] HS-FRI [22] HTY-FRI [53] HCL-FRI [42]	B^*=(5.33 6.33 9.00) B^*=(5.33 6.33 9.00) B^*=(- 0.00 -) B^*=(5.33 6.33 8.33) B^*=(5.71 6.28 8.16) B^*=(-) B^*=(5.33 6.55 9.00)
The Incircle-FRI		**B^*=(5.28 6.37 8.28)**

Note: the sign (-) indicates no clear evidence for the method to handle the case in the example

- Based on **Eq.(4.7)**, **Eq.(4.16)**, **Eq.(4.18)**, and **Eq.(4.19)**, we obtain the left fuzziness of the left triangular $TR_LPS_1 = 0$ and the right fuzziness the right triangular $TR_RPS_3 = 1.05$.
- Consequently, the Incircle-FRI gets conclusion as trapezoidal fuzzy set $B^* = [4.54\ 4.54\ 7.47\ 8.53$, according to **Eq.(4.22)**.
- **Fig. 103** (*see Appendix A.4*) shows the results of the FRI methods. The HCL-FRI [42] is unable to generate a conclusion in this case. The KH-FRI [3], [25], [39], the KHstabilized-FRI [5], the VKK-FRI [4], and the HTY-FRI [53] generate an abnormal trapezoidal fuzzy set. In contrast, the CCL-FRI [52], the HS-FRI [22], and the proposed Incircle-FRI generate CNF trapezoidal fuzzy sets.

Concerning ratios of the fuzziness of the two rules $(A_1 \Rightarrow B_1)$ *and* $(A_2 \Rightarrow B_2)$ obtained by **Eq.(4.13)** and **Eq.(4.14)**, and **Table 4**, we can see that :

- The left ratio fuzziness $RF.PS_1\ (A_1, B_1) = 0.5$, $RF.PS_1\ (A_2, B_2) = 1$, and the left ratio fuzziness $RF.PS_1\ (A^*, B^*)$ of the proposed Incircle-FRI and CCL-FRI are 0.64 and 0.69. Therefore, the proposed Incircle-FRI and CCL-FRI methods satisfy property 1, where *MIN*

$(RF.PS_1 (A, B)) \leq RF.PS_1(A^*, B^*) \leq MAX (RF.PS_1(A, B))$ *is equal* $0.5 \leq [Incircle(0.64), CCL(0.69)] \leq 1$.

- With the right ratio fuzziness, we can also see that the ratio $RF.PS_3 (A_1, B_1) = RF.PS_3 (A_2, B_2) = 1$, and $RF.PS_3 (A^*, B^*)$ of the proposed Incircle-FRI, HTY FRI, HS FRI, CCL FRI, KHstabilized-FRI, and KH-FRI methods are 1, 1.62, 0.71, 1, 1, and 1, respectively. We can see that the right ratio fuzziness $RF.PS_3 (A^*, B^*)$ of the proposed Incircle-FRI, CCL FRI, KHstabilized-FRI, and KH-FRI satisfy Property 2.

Based on the above, the fuzzy interpolative reasoning results of the proposed method are logically consistent in terms of Property 1 and Property 2.

Table 4. *Fuzzy Interpolative Reasoning Results of Example TP1*

Attribute Values	**Methods**	**Results of Fuzzy Interpolative reasoning**
A_1=[0 4 5 6] A_2=[11 12 13 14] B_1=[0 2 3 4] B_2=[10 11 12 13] A^*=[6 6 9 10]	KH-FRI [3], [25], [39] KHstabilized-FRI [5] VKK-FRI [4] CCL-FRI [52] HS-FRI [22] HTY-FRI [53] HCL-FRI [42]	B^*=(5.45 4.25 7.50 8.5) B^*=(5.45 4.25 7.50 8.5) B^*=(5.32 4.38 7.38 8.68) B^*=(4.25 4.25 7.5 8.5) B^*=(5.23 5.23 7.61 8.32) B^*=(4.98 7.44 6.44 8.06) B^*=(-)
The Incircle-FRI		**B^*=(4.54 4.54 7.47 8.53)**

Note: the sign (-) indicates no clear evidence for the method to handle the case in the example

Example TP2 [52], [59]:

- The rule antecedents and the observation are triangular fuzzy sets; the consequents are trapezoidal fuzzy sets.
- Observation $A^* = [7, 8, 9]$.

Table 5 and **Fig. 104** (*see Appendix A.5*) show all the results of the FRI methods and attributes values of this example, based on **Eq.(4.6)**, **Eq.(4.7)**, **Eq.(4.16)**, **Eq.(4.17)**, **Eq.(4.18)**, and **Eq.(4.19)** the conclusion of the Incircle-FRI can be calculated by the fuzzy interpolation of the fuzzy sets A_1, A_2, B_1, and B_2; therefore, we conclude the following:

- The reference points $GPx.B^*_L = 5.331$, $GPx.B^*_R = 6.435$, and $MP.B^* = 5.9$.
- The fuzziness values of the left triangular are $TR_LPS_1 = 0.3233$ and $TR_LPS_3 = 0.0$.
- The fuzziness values of the right triangular are $TR_RPS_1 = 0.0$ and $TR_RPS_3 = 0.898$.
- Therefore, the Incircle-FRI produced a trapezoidal fuzzy conclusion $B^* = [5.01\ 5.90\ 5.90\ 7.33]$, which is calculated by **Eq.(4.23)**.
- In this case, the HCL-FRI is unable to generate any conclusion. The conclusion of the KH-FRI [3], [25], [39], the KHstabilized-FRI [5], and the VKK-FRI [4] are not convex and

normal. The CCL-FRI [52], the HS-FRI [22], the HTY-FRI [53], and the Incircle-FRI generate CNF trapezoidal conclusion.

Based on by **Eq.(4.13)** and **Eq.(4.14)**, and **Table 5**, the ratios of fuzziness the two adjacent rules $(A_1 \Rightarrow B_1, A_2 \Rightarrow B_2)$, and the ratios of fuzziness of the observation and conclusion $(A^* \Rightarrow B^*)$ of the FRI methods.

- The left ratio fuzziness of $RF.PS_1\ (A_1, B_1) = 0.4$, $RF.PS_1\ (A_2, B_2) = 0.5$, and the left ratio fuzziness $RF.PS_1\ (A^*, B^*)$ of the proposed Incircle-FRI, the HS FRI, the CCL FRI, and HTY-FRI are 0.44, 0.43, 0.44, and 0.33, respectively, therefore, the proposed Incircle-FRI, the HS FRI, and the CCL-FRI satisfy property 1, where $MIN\ (RF.PS_1\ (A_1, B_1), RF.PS_1\ (A_2, B_2)) \leq RF.PS_1\ (A^*, B^*) \leq MAX\ (RF.PS_1\ (A_1, B_1), RF.PS_1\ (A_2, B_2))$, which is equal $0.4 \leq (0.44, 0.44, 0.43) \leq 0.5$.
- In the right ratio fuzziness, we can also see that the ratio $RF.PS_3\ (A_1, B_1) = RF.PS_3\ (A_2, B_2) = 1$, and $RF.PS_3\ (A^*, B^*)$ of the proposed Incircle-FRI, the HS FRI, the CCL FRI, HTY FRI, KHstabilized-FRI, and KH-FRI are 1, 0.67, 1, 0.21, 1, and 1. Therefore, the right ratio fuzziness $RF.PS_3\ (A^*, B^*)$ of the proposed Incircle-FRI, CCL FRI, KHstabilized-FRI, and KH-FRI satisfy Property 2.

Based on the above, the fuzzy interpolative reasoning results of the proposed method are logically consistent in terms of Property 1 and Property 2.

Table 5. *Fuzzy Interpolative Reasoning Results of Example TP2*

Attribute Values	**Methods**	**Results of Fuzzy Interpolative reasoning**
A_1=[0 5 6] A_2=[11 13 14] B_1=[0 2 3 4] B_2=[10 11 12 13] A^*=[7 8 9]	KH-FRI [3], [25],[39] KHstabilized-FRI [5] VKK-FRI [4] CCL-FRI [52] HS-FRI [22] HTY-FRI [53] HCL-FRI [42]	B^*=(6.36 5.38 6.38 7.38) B^*=(6.36 5.38 6.38 7.38) B^*=(6.16 5.38 6.38 7.84) B^*=(4.94 5.38 6.38 7.38) B^*=(5.93 6.36 6.80 7.47) B^*=(5.87 6.20 7.20 7.41) B^*=(-)
The Incircle-FRI		**B^*=(5.01 5.90 5.90 7.33)**

Note: the sign (-) indicates no clear evidence for the method to handle the case in the example

Example TP3 [52]:

- The rule antecedents and the observation are trapezoidal fuzzy sets; the consequents are triangular fuzzy sets.
- Observation $A^* = [6, 6, 9, 10]$.

Table 6 and **Fig. 105** (*see Appendix A.6*) show all the results of the FRI methods and attributes values to this example. The representative values of the proposed Incircle-FRI are determined by the fuzzy interpolation of the fuzzy sets A_1, A_2, B_1, and B_2 using **Eq.(4.6)**, **Eq.(4.7)**, **Eq.(4.16)**, **Eq.(4.17)**, **Eq.(4.18)**, and **Eq.(4.19)**. This case,

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- The reference points $GPx.B^*_L = 4.3748$ and $GPx.B^*_R = 6.6639$.
- The fuzziness values of the left triangular are $TR_LPS_1 = 0$ and $TR_LPS_3 = 0$.
- The fuzziness values of the right triangular are $TR_RPS_1 = 0$ and $TR_RPS_3 = 1.90$.
- The proposed Incircle-FRI produced a trapezoidal fuzzy conclusion $B^* = [4.37\ 4.37\ 6.66\ 8.57]$ that is calculated according to **Eq.(4.22)**.
- The KH-FRI [3], [25], [39], the KHstabilized-FRI [5], and the HTY-FRI [53] cannot generate a convex and normal fuzzy conclusion. Meanwhile, the CCL-FRI [52], the HS-FRI [22], and the VKK-FRI [4] have a CNF trapezoidal fuzzy conclusion.

Based on **Eq.(4.13)** and **Eq.(4.14)**, and **Table 6**, the left ratio fuzziness of the *Rule1* $(A_1 \Rightarrow B_1)$ $RF.PS_1 (A_1, B_1) = 0.5$ and the ratio of the *Rule2* $(A_2 \Rightarrow B_2)$ $RF.PS_1 (A_2, B_2) = 1$. Thus, we can see that:

- The left ratio fuzziness $RF.PS_1 (A^*, B^*)$ of the proposed Incircle-FRI and CCL-FRI are 0.64 and 0.69. Therefore, the proposed Incircle-FRI and CCL-FRI satisfy property 1, where $MIN\ (RF.PS_1 (A_1, B_1), RF.PS_1 (A_2, B_2)) = 0.5 \leq RF.PS_1 (A^*, B^*)\ [Incircle\ (0.64), CCL\ (0.69)] \leq MAX\ (RF.PS_1 (A_1, B_1), RF.PS_1 (A_2, B_2)) = 1$.
- For the right ratio fuzziness, the ratio $RF.PS_3 (A_1, B_1) = RF.PS_3 (A_2, B_2) = 2$, and the right ratio fuzziness $RF.PS_3 (A^*, B^*)$ of the proposed Incircle-FRI, CCL FRI, the HTY FRI, HS FRI, the VKK-FRI, the KHstabilized-FRI, and the KH-FRI methods are 2, 2, 2.8, 3.1, 3.3, 2, and 2, respectively. We can see that only $RF.PS_3 (A^*, B^*)$ of the proposed Incircle-FRI, CCL FRI, the KHstabilized-FRI, and the KH-FRI satisfy Property 2, in which $RF.PS_3 (A_1, B_1) = RF.PS_3 (A_2, B_2) = RF.PS_3 (A^*, B^*) = 2$.

Based on the above, the fuzzy interpolative reasoning result of the proposed method is logically consistent in terms of Property 1 and Property 2.

Table 6. *Fuzzy Interpolative Reasoning Results of Example TP3*

Attribute Values	Methods	Results of Fuzzy Interpolative reasoning
A_1=[0 4 5 6] A_2=[11 12 13 14] B_1=[0 2 4] B_2=[10 11 13] A^*=[6 6 9 10]	KH-FRI [3],[25],[39] KHstabilized-FRI [5] VKK-FRI [4] CCL-FRI [52] HS-FRI [22] HTY-FRI [53] HCL-FRI [42]	B^*=(5.45 4.25 6.50 8.50) B^*=(5.45 4.25 6.50 8.50) B^*=(5.31 5.38 5.38 8.68) B^*=(5.38 5.38 5.38 7.38) B^*=(5.46 5.46 5.46 8.55) B^*=(5.07 7.26 5.26 8.15) B^*=(-)
The Incircle-FRI		**B^*=(4.37 4.37 6.66 8.57)**

Note: the sign (-) indicates no clear evidence for the method to handle the case in the example

Table 7 presents a summary of evaluation for the proposed Incircle-FRI method compared with the current methods (i.e., KH-FRI [3], [25], [39], KHstabilized-FRI [5], VKK-FRI [4], CCL-FRI [52], HS-FRI [22], HTY-FRI [53], and HCL-FRI [42]) according to the criteria (i.e., "CNF property", "different membership functions", which mean there is no restriction the shape of the fuzzy sets, "different kinds membership functions of the antecedents and the consequents fuzzy rules", "the Approximation capability of the fuzziness and core between the observation and conclusion", and "logically consistent with respect to the ratios of fuzziness sides (see property 1 and property 2)").

Table 7. *Description of the Evaluation Criteria of the Incircle-FRI Method with Existing Methods*

Criteria	Methods							
	KH FRI [1]-[3]	KHstabilized-FRI [5]	VKK FRI [4]	HCL FRI [12]	HTY FRI [15]	CCL FRI [14]	HS FRI [13]	Incircle -FRI
CNF property	x	x	x	-	-	√	√	√
Handle Different Membership Function	√	√	√	x	-	√	√	√
The ANT and the CON membership functions can be different	√	√	√	x	-	√	√	√
The Approximation capability of the fuzziness and core between observation and Conclusion	x	x	x	x	x	-	-	√
Logically consistent with respect to the ratios of fuzziness	-	-	x	x	x	√	-	√

From **Table 7** we can see that the Incircle-FRI methods satisfy with these five evaluation criteria, where the sign (√) indicates the technique is satisfied with all criteria for all selected examples, while a sign (-) shows the method has a problem in most examples, and the sign (x) indicates the technique does not satisfy with all examples.

SUMMARY

In this chapter, a new fuzzy interpolative reasoning method called "*Incircle-FRI*", is introduced, which is defined for triangular CNF fuzzy sets, for a single antecedent universe and two surrounding rules from the rule-base. The proposed "*Incircle-FRI*" is based on the incircle of a triangular fuzzy number, the Gergonne Point (*GP*) as a reference point of the triangular fuzzy set, and the "*fuzziness sides*", i.e., the distances of the endpoints of the support, and the core from the incircle InTouch points (noted by PS_1, PS_2, and PS_3 in this chapter).

The proposed method calculates the conclusion by holding the same rate of distances among the observation and the two rule antecedents, and the conclusion and the two corresponding rule consequents with the Gergonne Points (for the reference point of the conclusion), and with the "fuzziness sides" (for the shape of the conclusion). The chapter also extends the suggested "Incircle-FRI" to trapezoidal-shaped fuzzy sets by decomposing their shapes to multiple triangular. The generated conclusions are always a CNF fuzzy set for triangular and trapezoidal fuzzy sets. The performance of the proposed "Incircle-FRI" is discussed based on numerical examples, and a comprehensive comparison to other FRI methods, namely with the (i.e., KH-FRI [3], [25], [39], KHstabilized-FRI [5], VKK-FRI [4], CCL-FRI [52], HS-FRI [22], HTY-FRI [53], and HCL-FRI [42]). From the experimental results and **Table 7**, we can see that the proposed method is considered one of the best current FRI methods. Consequently, the proposed method provides a useful method as a fuzzy interpolation in dispersed rules-based systems.

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INTRODUCTION

In this chapter, we will discuss the extensions of the Incircle-FRI for the hexagonal membership function, multidimensional antecedent variables, and extrapolation using the shifting ratio and modification of the weighting. Additionally, to prove the performance extensions of the "Incircle-FRI" method, some numerical examples will be used to compare Incircle-FRI with the existing FRI methods (to KH-FRI [3], [25], [39], KHstabilized-FRI [5], VKK-FRI [4], CCL-FRI [52], HS-FRI [22], HTY-FRI [53], HCL-FRI [42], MACI-FRI [6], IMUL-FRI [8], and CRF-FRI [7].

5.1. Extensions of the Incircle-FRI Method

Traditional fuzzy reasoning methods demand a complete fuzzy rule base to conclude a result, but due to incomplete data or lack of knowledge, complete rule bases are not always available. Besides, many interpolation methods presume that the two closest adjacent rules to the observation are available, and flank the observation for each attribute (but not necessarily in the same order). In practice, however, there may be a different number of the closest rules to a given observation, and the attribute values of these rules may lie just on one side of the observation. Some interpolation methods cannot handle cases where fuzzy sets with crisp borders are involved. These limitations inevitably restrict the potential application of some of the existing FRI techniques. Although fuzzy interpolation has been applied to control problems [95], [96], [97], relatively few application examples exist in the area of prediction and classification.

The proposed "Incircle-FRI" introduced in (Chapter 4) can only handle triangular and trapezoidal fuzzy sets. In this chapter, the hexagonal membership functions will be discussed, and also to allow interpolation that requires multiple-antecedent rules. For this reason, we suggested a modification of the weight estimate and included a shift technique in order to ensure that the reference point (GP) of the observation and the reference point (GP) of the interpolated (intermediate) observation are mapped together. This weight calculation and shift technique ensure more reasonably to interpolate the consequent fuzzy result, and this will also enhance the capability for extrapolation. It is shown that exploiting the generality of these extensions, and extrapolation can be performed over multiple-antecedent rules in a straightforward manner.

5.1.1. Extension of the Incircle-FRI to single antecedent with hexagonal fuzzy set

The Incircle concept of a triangular fuzzy number can extend to hexagonal or any complex polygonal fuzzy membership functions. A hexagonal fuzzy set based on the Incircle-FRI can be represented by two triangular fuzzy sets $AL= (a_1, a_2, a_3)$ and $AR= (a_4, a_5, a_6)$. **Fig. 51** describes the hexagonal fuzzy set representative values, denoted by $(a_1, a_2, a_3, a_4, a_5, a_6)$. a_3 and

a_4 denote the left and right reference points, respectively, a_1 and a_6 refer to the left and the right sides of the support points, respectively, a_2 and a_5 denote the intermediate points.

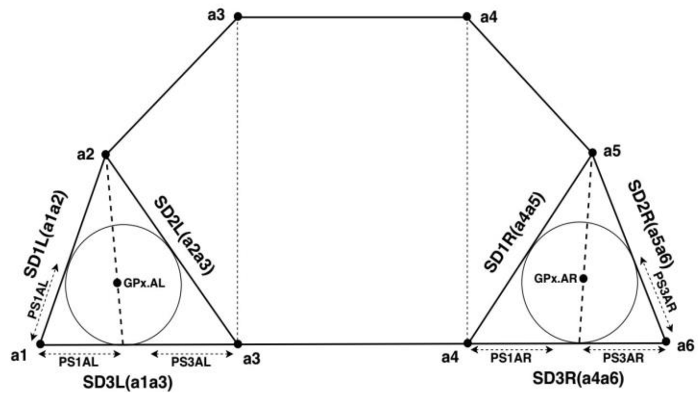

Fig. 51. *The Main Incircle_Notations of the Hexagonal Fuzzy Number Represented by Two Triangular Fuzzy Sets AL and AR Notations.*

Fig. 52 describes an example of the suggested Incircle fuzzy interpolative reasoning using the hexagonal fuzzy set.

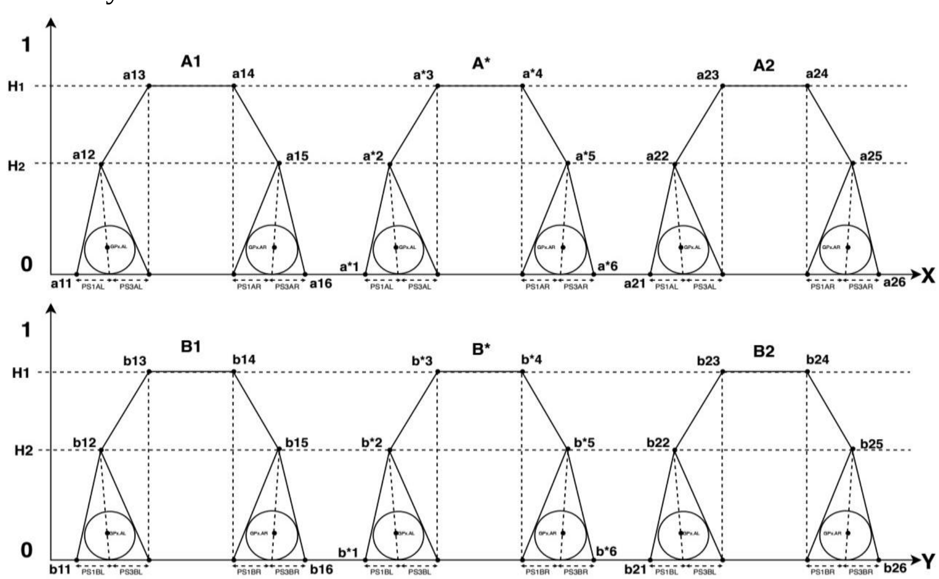

Fig. 52. *Fuzzy Interpolative Reasoning using Hexagonal Membership Functions*

In the following, we will present the main steps that will be used to interpolate the proposed Incircle-FRI reasoning method in case the Hexagonal fuzzy sets, which represented by two *AL* and *AR* triangular fuzzy sets, **Eq.(4.4)**, **Eq.(4.6)** and **Eq.(4.7)** could be used to calculate Incircle_Notations of AL and AR triangulars separately, then applying the following steps:

Step 1: The two closest fuzzy rules [56] must be determined to perform interpolative for single antecedent fuzzy rules (m), $Rule_1, Rule_2, \ldots, and\ Rule_m$, which are the nearest to the observation

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A^* as shown in **Fig. 52**. The closest fuzzy rules determined using the average of two reference points *GPx.AL* and *GPx.AR* of the hexagonal fuzzy set by $AVG.GP_X=(GP_X.AL \text{ and } GP_X.AR)/2$. Then, distances can be computed using **Eq.(5.1)**.

$$D = d(A_i, A^*) = d(AVG.GP_x.A_i, AVG.GP_x.A^*) \tag{5.1}$$

Step 2: Supposing that two adjacent fuzzy rules A_1 and A_2 are the left and right antecedent fuzzy sets to the observation fuzzy set A^*. Hence, the weight of hexagonal fuzzy rules neighboring and observation can be determined for each triangular (the left and right) by **Eq.(5.2)**.

$$WL_i = 1 - \frac{\left|GP_x.A^*L - GP_x.AL_i\right|}{GP_x.A_2L - GP_x.A_1L}, \qquad WR_i = 1 - \frac{\left|GP_x.A^*R - GP_x.AR_i\right|}{GP_x.A_2R - GP_x.A_1R} \tag{5.2}$$

Step 3: The two reference points $GP_X.AL$ and $GP_X.AR$ of the fuzzy interpolative reasoning result B^* could be calculated for left and right triangular by **Eq.(5.3)**.

$$GP_x.B^*L = \sum_{i=1}^{2} WL_i \times GP_x.BL_i, \qquad GP_x.B^*R = \sum_{i=1}^{2} WR_i \times GP_x.BR_i \tag{5.3}$$

Step 4: The fuzziness sides of the B^* fuzzy set can be calculated for the left triangular by **Eq.(5.4)** and the right triangular by **Eq.(5.5)**.

$$PS_M(BL^*) = \begin{Bmatrix} PS_M(AL^*) \times \sum_{i=1}^{2} WL_i \times \frac{PS_M(BL_i)}{PS_M(AL_i)}, \\ \\ if\ \exists_i\ PS_M(AL_i) > 0 \\ PS_M(AL^*), \\ if\ \forall_i\ PS_M(AL_i) = 0 \end{Bmatrix} \tag{5.4}$$

$$PS_M(BR^*) = \begin{Bmatrix} PS_M(AR^*) \times \sum_{i=1}^{2} WR_i \times \frac{PS_M(BR_i)}{PS_M(AR_i)}, \\ \\ if\ \exists_i\ PS_M(AR_i) > 0 \\ PS_M(AR^*), \\ if\ \forall_i\ PS_M(AR_i) = 0 \end{Bmatrix} \tag{5.5}$$

Step 5: Based on the results of steps 3 and 4, the reference points of the conclusion ($GP_X.BL^*$ and $GP_X.BR^*$), and the fuzziness sides (PS_1 and PS_3) could be used to determine the conclusion B^* of the hexagonal fuzzy set. Finally, the conclusion B^* of the Incircle fuzzy interpolative reasoning for the hexagonal fuzzy set can be determined by **Eq.(5.6)**.

$$
\begin{aligned}
B_1^* &= GP_x.B^*L - B^*L_{(PS1)} \\
B_2^* &= GP_x.B^*L \\
B_3^* &= GP_x.B^*L + B^*L_{(PS3)} \\
B_4^* &= GP_x.B^*R - B^*R_{(PS1)} \\
B_5^* &= GP_x.B^*R \\
B_6^* &= GP_x.B^*R + B^*R_{(PS3)}
\end{aligned}
\tag{5.6}
$$

Because $(B^*_1 \leq B^*_2 \leq B^*_3 \leq B^*_4 \leq B^*_5 \leq B^*_6)$, we can see that the proposed method can preserve the convexity of the fuzzy interpolative reasoning result with a hexagonal fuzzy set.

5.1.2. Extension of the Incircle-FRI to multiple fuzzy rules and multiple antecedent variables.

The case described in Chapter 4 concerns interpolation between two adjacent rules, each involving one antecedent variable; this means it is easily extendable to rules with multiple antecedent variables. Of course, the variables determined in both rules must be the same to make sense for interpolation. Therefore, in the following, Incircle-FRI with multiple fuzzy rules and multiple antecedent fuzzy interpolative reasoning will be discussed.

The reasoning scheme below is an example with *n* number of rules and *m* number of antecedents.

***R**₁*: *If* X_1 *is* A_{11} *and* X_2 *is* A_{12} ……… *and* X_m *is* A_{1m} *Then Y is* B_1;
***R**₂*: *If* X_1 *is* A_{21} *and* X_2 *is* A_{22} ……… *and* X_m *is* A_{2m} *Then Y is* B_2;
……………………………………………………………………….
$\boldsymbol{R_n}$: *If* X_m *is* A_{nm} *and* X_2 *is* A_{nm} ……… *and* X_m *is* A_{nm} *Then Y is* B_n;
Observation: *If* X_1 *is* A_1^* *and* X_2 *is* A_2^* ……… *and* X_m *is* A_n^*;

Conclusion: *Y is* B^*

Let us assume that the observation A_1^* is flanked by four rules with two antecedents rules on each side; namely A_{11} and A_{21} on the left, and A_{31} and A_{41} on the right of the first antecedent, and A_2^* is flanked by four rules with two rules on each side; namely A_{12} and A_{22} on the left, and A_{32} and A_{42} on the right of the second antecedent. The four adjacent fuzzy rules are as follows, $(A_{11} \wedge A_{12} \rightarrow B_1, A_{21} \wedge A_{22} \rightarrow B_2, A_{31} \wedge A_{32} \rightarrow B_3, \text{ and } A_{41} \wedge A_{42} \rightarrow B_4)$.

An example of the Incircle-FRI reasoning using multiple fuzzy rules and multiple antecedents described by trapezoidal fuzzy sets is shown in **Fig. 53**.

The multiple antecedent fuzzy sets A_{ij} could be denoted by trapezoidal fuzzy set (a_{ij1}, a_{ij2}, a_{ij3}, a_{ij4}), which can be represented through two triangular fuzzy sets $AL_{ij} = (a_{ij1}, a_{ij2}, a_{ijMp}; H)$ and $AR_{ij} = (a_{ijMp}, a_{ij3}, a_{ij4}; H)$, the observation fuzzy set A_j^* is denoted by $(a_1^*, a_2^*, a_3^*, a_4^*)$ $((AL_j^* = (a_1^*, a_2^*, a_{Mp}^*; H)$ and $AR_j^* = (a_{Mp}^*, a_3^*, a_4^*; H))$, the consequent fuzzy set B_i is denoted by $(b_{i1},$

b_{i2}, b_{i3}, b_{i4}) ($BL_i = (b_{i1}, b_{i2}, a_{ijMp}; H)$ and $BR_i = (a_{ijMp}, b_{i3}, b_{i4}; H)$), where $i = 1, 2, .., n$ and $j = 1, 2, ..., m$ and the fuzzy interpolative reasoning result B^* is denoted by $(B_1^*, B_2^*, B_3^*, B_4^*)$.

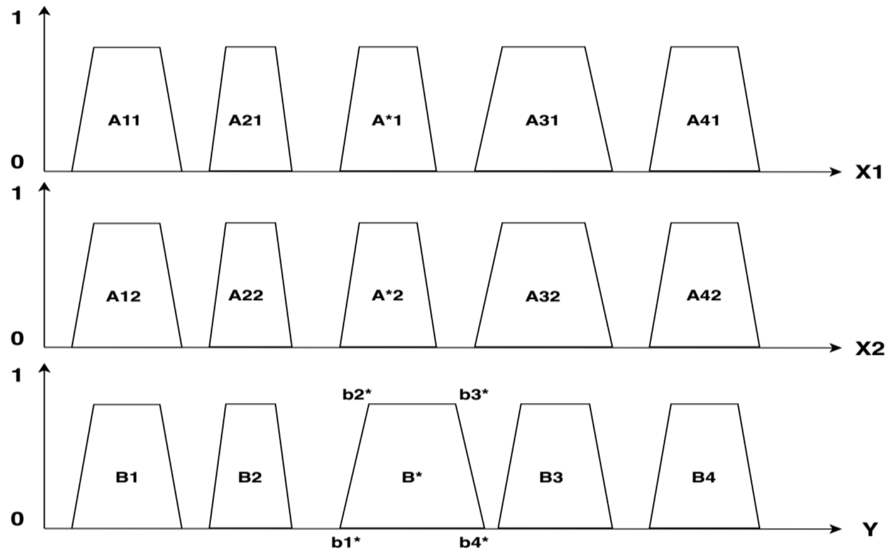

Fig. 53. *Fuzzy Interpolative Reasoning Using Trapezoidal Membership Functions*

To interpolate the proposed Incircle fuzzy interpolative reasoning method with multiple fuzzy rules and multiple antecedents using trapezoidal fuzzy sets (represented by two *AL* and *AR* triangular fuzzy sets), we need to find all Incircle_Notations in **Eq.(4.4) - Eq.(4.7)** for each triangular (*AL*, *AR*), then applying the following steps:

Step 1: The two closest fuzzy rules [56] must be determined to perform interpolative for single antecedent fuzzy rules (m), $Rule_1, Rule_2, \ldots, and\ Rule_n$, which are the nearest to the observation ($A_{ij} \preccurlyeq A_i^* \preccurlyeq A_{ij+1}$), as shown in **Fig. 53**. The closest fuzzy rules determined using the average of two reference points $GPx.AL_{ij}$ and $GPx.AR_{ij}$ of the trapezoidal fuzzy sets via ($AVG.GP_X.A = (GP_X.AL_{ij}\ and\ GP_X.AR_{ij})/2$). Then, distances can be computed using **Eq.(5.7)**.

$$D = d(A_{ij}, A_i^*) = d(AVG.GP_x.A_{ij}, AVG.GP_x.A_i^*) \tag{5.7}$$

Step 2: The weight calculation between trapezoidal fuzzy rules neighboring and observation could be determined by **Eq.(5.8)** and **Eq.(5.9)**. The reference point of the intermediate conclusion $B`$ can be determined by **Eq.(5.10)**.

$$w(s)_{ij} = 1 - \frac{\left|GP_x.A_j^*(s) - GP_x.A(s)_{ij}\right|}{GP_x.A_{nj}(s) - GP_x.A_{1j}(s)}, \tag{5.8}$$

$$W(s)_i = \frac{\sum_{j=1}^{m} w(s)_{ij}}{\sum_{i=1}^{n}\sum_{j=1}^{m} w(s)_{ij}} \tag{5.9}$$

$$GP.B'_{(s)} = \sum_{i=1}^{n} W(s)_i \, GPx.B_{(s)_i} \tag{5.10}$$

where $(s) \in [L, R]$ denote the weight of left and right of the Rulei, $(0 \leq WL_{ij} \leq 1)$, $(0 \leq WR_{ij} \leq 1)$, and $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, n denotes the number of rules and m denotes the number of antecedent variables. In the example given for illustration in this subsection, $n = 4$ and $m = 2$ as shown in **Fig. 53**. Holding the property of $(WL_{1j} + WL_{2j} = 1)$, $(WR_{1j} + WR_{2j} = 1)$, *and* $(W1 + W2 = 1)$.

Step 3: The two reference points $GP_X.AL$ and $GP_X.AR$ of the fuzzy interpolative reasoning result B^* could be calculated for left and right triangular by **Eq.(5.11)**.

$$GP_x.B^*L = \sum_{i=1}^{n} WL_i \times GP_x.BL_i, \qquad GP_x.B^*R = \sum_{i=1}^{n} WR_i \times GP_x.BR_i \tag{5.11}$$

Step 4: The fuzziness sides of the B^* fuzzy set can be calculated for the left triangular by **Eq.(5.12)** and for the right triangular by **Eq.(5.13)**.

$$PS_M(BL^*) = \begin{cases} \sum_{j=1}^{m} PS_M(AL^*) \times \sum_{\substack{i=1 \\ \exists ij PS_M(ALij)>0}}^{n} WL_i \times \frac{PS_M(BL_{ij})}{\sum_{j=1}^{m} PS_M(AL_{ij})}, \\ \quad if \;\; \exists_{ij} PS_M(AL_{ij}) > 0 \\ \frac{\sum_{j=1}^{m} PS_M(AL_j{}^*)}{m}, \\ \quad if \;\; \forall_{ij} PS_M(AL_{ij}) = 0 \end{cases} \tag{5.12}$$

$$PS_M(BR^*) = \begin{cases} \sum_{j=1}^{m} PS_M(AR^*) \times \sum_{\substack{i=1 \\ \exists ij PS_M(ARij)>0}}^{n} WR_i \times \frac{PS_M(BR_{ij})}{\sum_{j=1}^{m} PS_M(AR_{ij})}, \\ \quad if \;\; \exists_{ij} PS_M(AR_{ij}) > 0 \\ \frac{\sum_{j=1}^{m} PS_M(AR_j{}^*)}{m}, \\ \quad if \;\; \forall_{ij} PS_M(AR_{ij}) = 0 \end{cases} \tag{5.13}$$

$M \in [PS_1, PS_3]$, PS_1 refers to the left fuzziness side and PS_3 denotes the right fuzziness side for the two triangle fuzzy sets *AL* and *AR*. If PS_1 or PS_3 of A_1 and A_2 fuzzy sets are bigger than zero, then the top part of **Eq.(5.12)** and **Eq.(5.13)** could be used to conclude the left fuzziness side PS_1 and the right fuzziness side PS_3 of the fuzziness of B^* fuzzy set, if both PS_1 or PS_3 of A_1 and A_2 fuzzy sets are zero, the bottom part of **Eq.(5.12)** and **Eq.(5.13)** could be used.

Step 5: Based on the results of step 3 and 4, the reference points ($GP_X.BL^*$ and $GP_X.BR^*$), and the fuzziness sides (PS_1 and PS_3) could be used to determine the conclusion B^* of the trapezoidal fuzzy set, which based on the following cases:

1. **In case** the results of the left and right B^*TR_L and $B^*\ TR_R$ triangulars fuzzy sets have the same values; then, the conclusion can be determined by **Eq.(5.14)**.

$$B^* = [GP_X.BL^*, GP_X.BL^*, GP_X.BR^*, GP_X.BR^*] \tag{5.14}$$

2. **In case** the result of the left triangular B^*TR_L has the same values, and the right triangular B^*TR_R has the same values too, but both left and right values are not the same. The conclusion values could be defined by ($GP_{x1x2} = (GP_X.BL^* + GP_X.BR^*)/2$) as follows.

$$B^* = [GP_{x1x2}, GP_{x1x2}, GP_{x1x2}, GP_{x1x2}] \tag{5.15}$$

3. **In case** the left has the same values, we will use B^*L, but in case the right has the same values, we will use B^*R, then the conclusion can be determined by **Eq.(5.16)**.

$$\begin{aligned} BL^* &= [GP_x.B^*L, GP_x.B^*L, GP_x.B^*R, GP_x.B^*R+B^*R_{(ps3)}] \\ BR^* &= [GP_x.B^*L-B^*L_{(ps1)}, GP_x.B^*L, GP_x.B^*R, GP_x.B^*R] \end{aligned} \tag{5.16}$$

4. **In case** all values result from the left and the right triangles are different. The conclusion can be determined by **Eq.(5.17)**.

$$\begin{aligned} B_1^* &= GP_x.B^*L - B^*L_{(PS1)} \\ B_2^* &= GP_x.B^*L \\ B_3^* &= GP_x.B^*R \\ B_4^* &= GP_x.B^*R + B^*R_{(PS3)} \end{aligned} \tag{5.17}$$

5.1.3. Extension of the Incircle-FRI to extrapolation using shift ratio and weight measurement

The Incircle-FRI method is constructed to perform "Interpolation". Still, it cannot handle "Extrapolation" mainly due to two factors, which are the weight derivation and the lack of shift

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in a fuzzy set. In the following, we will debate the modification in weight derivation and introduce the shifting process.

5.1.3.1. Extended weight computation

To extend the Incircle-FRI method to be usable in extrapolation, the weight computation must be extended. The weight computation, as defined **Eq.(5.9)** is suitable for two adjacent single antecedent fuzzy rules, but it is unsuitable for handling more rules and multiple antecedents. To solve this issue, in [57], extend weight computation is suggested for more rules. The weight computation not only for the closest rules but also gives adequate weights to furthest rules.

Fig. 54 represents an example of the various antecedent fuzzy sets with different distances, $dis_{(1)}=4, dis_{(2)}= 7, dis_{(3)}=10, dis_{(4)}=3, dis_{(5)}=6, dis_{(6)}=36, dis_{(7)}=40$. To explain that, let fuzzy set A_2 be the observation, and the rest of the fuzzy sets be neighboring rules. Thus, the current implementation will take the distance between fuzzy set A_1 and A_4, which is $dis_{(3)}$ as the denominator for **Eq.(5.8)**.

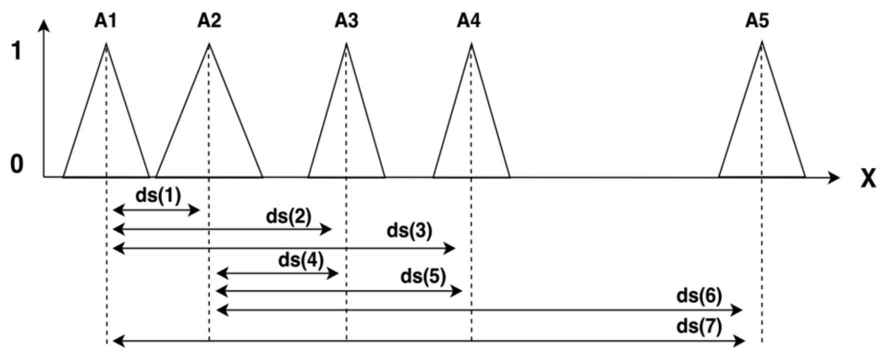

Fig. 54. *Fuzzy Sets Distance Measure*

The extended weight computation, an overall distance measure, was considered for the divisor as in given **Eq.(5.18)**. The denominator of the equation sums up all distances between the observation and rules. Normalization on the weights must be performed to each antecedent domain, as presented in **Eq.(5.19)**. This method holds the respective value of the weight, and it even shows that the farthest distance of rules does not influence the weight distribution.

$$w_{ij} = 1 - \left| \frac{GPa_j^* - GPa_{ij}}{\sum_{i=1}^{n} \left| GPa_j^* - GPa_{ij} \right|} \right|, \tag{5.18}$$

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$$W_{ij} = \frac{w_{ij}}{\sum_{i=1}^{n} w_{ij}}, \tag{5.19}$$

For more details about the difference between the original and extended weight, in [57] a comparison between them was determined, for example, the weight assigned for furthest rule from the observation is higher compared to the original method, in the case of using four rules, the weight between A_2A_5 fuzzy sets is 0.036. While, in extended weight, weight is more shared towards the furthest rule, where A_2A_5 is 0.088.

Let us take an extrapolation example (see **Fig. 54)**, the fuzzy set A_1 is the observation and fuzzy set A_2, A_3, and A_4 are the three nearest rule. The distance between all fuzzy sets remains the same. In this example, the extreme rules distance is represented by $dis_{(5)}$. When computing the weight value for W_{A1A3} and W_{A1A4} using **Eq.(5.9)**, the negative weight value will arise because the numerator is larger than the denominator. This problem can be resolved using **Eq.(5.19)** instead, which will also assist in establishing the extrapolation capability, which will be further discussed in the following.

5.1.3.2. Extended Shift Ratio

The reference point (GP) of the rule sets can be successfully applied to interpolate the consequent if the only two most adjacent rules are considered. By implementing the weights obtained from **Eq.(4.9)**, an interpolated observation can be calculated at the same (GP) as the real observation. This guarantees that the fuzzy conclusion is interpolated the distance of (GP) of the observation and rules' antecedent (see, e.g., in **Fig. 47**).

Nevertheless, when more than two rules are included, the intermediate observation, GpA', does not share the same (GP) as the observation, A^*, as shown in **Fig. 55**. GpA' is derived from extended weight, where $W_{A1A2} = 0.346$, $W_{A2A3} = 0.384$, and $W_{A2A4} = 0.269$ with their reference point (GP) at e.g., $A_1 = 3$, $A_2 = 8$, $A_3 = 10$, and $A^* = 5$. GpA'_j is determined by **Eq.(5.20)**:

$$GPA'_j = \sum_{i=1}^{n} w_{ij} GPA_{ij}, \tag{5.20}$$

where $i = 1, 2, \ldots, n$ refer to the number of rules, $j = 1, 2, \ldots, m$ refer to the number of antecedents and GpA'_j . In the above example, GpA' is computed, which is 5.648. A shift of GpA' to GpA^* is required to align the intermediate observation to be the same reference point (GP) of the original observation.

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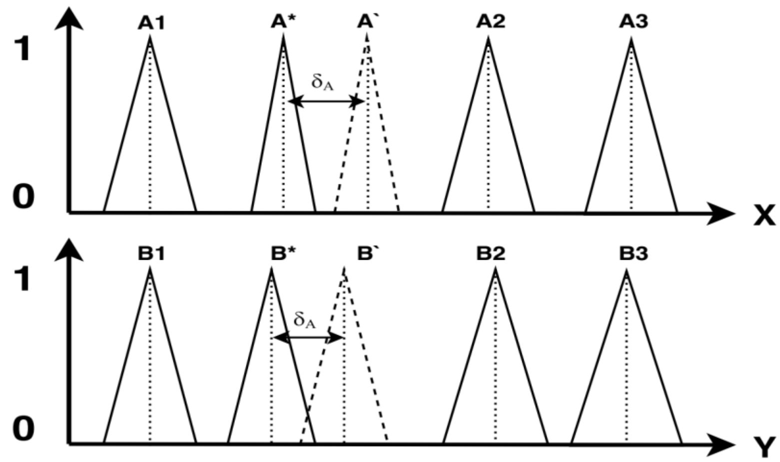

Fig. 55. *Interpolation Involving Three Rules*

A shift ratio can be derived using the distance between (*GP*) of the observation and the intermediate fuzzy set. The shift ratio δ_A can be calculated by **Eq.(5.21)**:

$$\delta_A = \frac{\sum_{j=1}^{m} \delta_{Aj}}{m}, \tag{5.21}$$

Where

$$\delta_{Aj} = \frac{GPA_j^* - GPA_j'}{\left|GPA_{nj} - GPA_{1j}\right|}. \tag{5.22}$$

The reference point (*GP*) of the last fuzzy rule in the antecedent is indicated by GPA_{nj}, where $j = 1, 2, \ldots, m$ denotes the number of antecedents. **Eq.(5.22)** computes the shift ratio across an antecedent domain using (*GP*) distance between the observation and the intermediate rule concerning (*GP*) distance of the first and the last rule. As shown in **Eq.(5.21)** is the average of the result of **Eq.(5.22)** from all antecedents fuzzy sets.

The reference point of the fuzzy conclusion, $GP_X.B^*$, is next calculated by the shift ratio δ_A and the intermediate consequent from **Eq.(5.10)** as b' using the following:

$$GPx.B^* = GPx.B' + \delta_A(\left|GPx.B_n - GPx.B_1\right|). \tag{5.23}$$

In our example (see **Fig. 55**), the reference point of the fuzzy conclusion $GP_X.B^*$ is 4.615. This shift ratio implementation enables the possibility of extrapolation. Let us consider the same example, there are only two rules, $A_2 \rightarrow B_2$ and $A_3 \rightarrow B_3$, with observation A^* to show an extrapolation example. Using **Eq.(5.19)**, we get $W_{A*A2} = 0.60$ and $W_{A*A3} = 0.40, GpA' = 9.6$ from

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Eq.(5.20), and with **Eq.(5.10)** we have $GPx.B' = 8.20$. The shift ratio δ_A is -1.512 computed from **Eq.(5.21)**, the difference between consequents (B_3 - B_2) fuzzy sets is 3, and the extrapolated reference point is $GP_X.B^* = 3.66$ by **Eq.(5.23)**.

5.2. Experimental results

In this subsection, several experiments and comparisons are conducted based on the extensions of the Incircle-FRI in comparison with different FRI methods. Firstly, we present one example, which aims to prove the performance of the Incircle-FRI with the hexagonal fuzzy set according to the results of the FRI methods (KH-FRI [3], [25], [39], KHstabilized-FRI [5], VKK-FRI [4], CCL-FRI [52], HS-FRI [22], HTY-FRI [53], HCL-FRI [42]). Secondly, we present two different examples (using triangular and trapezoidal fuzzy sets) aim to prove the validity of the proposed method in case multi antecedent variables comparing to the results of the FRI methods (KH-FRI [3], [25], [39], KHstabilized-FRI [5], VKK-FRI [4], CCL-FRI [52], HS-FRI [22], HTY-FRI [53], HCL-FRI [42], MACI-FRI [6], IMUL-FRI [8], and CRF-FRI [7]). Thirdly, we introduce two different examples (using single and multi-antecedents variables) to check the performance of the Incircle-FRI with extrapolation property with the FRI methods (KH-FRI [3], [25], [39], KHstabilized-FRI [5], VKK-FRI [4], HTY-FRI [53], HCL-FRI [42], MACI-FRI [6], IMUL-FRI [8], and CRF-FRI [7]). The KH-FRI, the KHstabilized-FRI, VKK-FRI, the MACI-FRI, the IMUL-FRI, and the CRF-FRI methods were tested by the Matlab FRI toolbox [24], [26].

5.2.1. Incircle-FRI with Hexagonal fuzzy set

Example HEX1 [22], [58], [59]:

- All the rule antecedents, consequents, and the observation are hexagonal fuzzy sets.
- The observation describes by hexagonal fuzzy set $A^* = [6, 6.5, 7, 9, 10, 10.5]$.

Table 8 and **Fig. 106** (see *Appendix A.7*

) show all the results of the FRI methods and attributes values to this example as follows:

- The representative values of the Incircle-FRI are determined by the fuzzy interpolation of the fuzzy sets A^*, A_1, A_2, B_1, and B_2 using **Eq.(4.6)**, **Eq.(4.7)**, **Eq.(5.2)**, **Eq.(5.3)**, **Eq.(5.4)**, and **Eq.(5.5)** that produce a hexagonal fuzzy conclusion $B^* = [5.59\ 6.02\ 6.55\ 8.63\ 9.52\ 10.13]$.
- By **Eq.(5.3)**, we determined the reference points of the Left triangular $GP_x.B^*_L = 6.017$ and Right triangular $GP_x.B^*_R = 9.52$.
- By **Eq.(5.4)** and **Eq.(5.5)**, the Left and Right fuzziness sides of the left triangular value are calculated, where ($TR_LPS_1 = 0.424$, $TR_LPS_3 = 0.531$), the Left and Right fuzziness sides values of the right triangular are ($TR_RPS_1 = 0.898$, $TR_RPS_3 = 0.601$).
- The results of the suggested methods the KH-FRI [3], [25], [39], the VKK-FRI [4], The CCL-FRI [52], and the HS-FRI [22] generate CNF hexagonal fuzzy conclusion. In

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contrast, the KHstabilized-FRI [5] cannot produce a CNF conclusion, the HTY-FRI [53], and HCL-FRI [42] have no conclusion at all.

Table 8. *Fuzzy Interpolative Reasoning Results of Example HEX1*

Attribute Values	Methods	Results of Fuzzy Interpolative reasoning
A_1=[0 1 3 4 5 5.5] A_2=[11 11.5 12 13 13.5 14] B_1=[0 0.5 1 3 4 4.5] B_2=[10.5 11 12 13 13.5 14] A^*=[6 6.5 7 9 10 10.5]	KH-FRI [3],[25],[39] KHstabilized-FRI [5] VKK-FRI [4] CCL-FRI [52] HS-FRI [22] HTY-FRI [53] HCL-FRI [42]	B^*=(5.73 5.74 5.75 5.89 7.25 8.56) B^*=(5.73 5.87 6.00 5.89 7.25 8.56) B^*=(5.60 5.69 5.71 6.26 7.46 8.68) B^*=(5.2 5.4 5.7 8.6 9.7 10.2) B^*=(5.47 5.79 6.08 8.42 9.23 9.70) B^*=(-) B^*=(-)
The Incircle-FRI		**B^*=(5.59 6.02 6.55 8.63 9.52 10.13)**

Note: the sign (-) indicates no clear evidence for the method to handle the case in the example

5.2.2. Incircle-FRI with multiple fuzzy rules having multiple antecedents

Example MultiA1 [52]:

- The Incircle-FRI with multiple antecedent variables where the fuzzy rules are $A_{11} \wedge A_{12} \Rightarrow B_1$, $A_{21} \wedge A_{22} \Rightarrow B_2$, and the observations A_1^* and A_2^* are given to determine the consequence B^*. All the rule antecedents, consequents, and the observation are trapezoidal fuzzy sets.
- Observation: A^*_1= [6 7 9 11], A^*_2= [6 8 10 12].
- By **Eq.(5.20)**, we derive the intermediate observation of the first antecedent A'_1 is (Left triangular $AL.A'_1=7.5$ and Right triangular $AR.A'_1=7.28$) and for the second antecedent A'_2 is (Left triangular $AL.A'_2=9.32$ and Right triangular $AR.A'_2=9.50$).
- Using the derived intermediate reference points (GP) A'_1, and A'_2, the computed shift ratio from each antecedent domain are determined by **Eq.(5.22)**, where δA_1 is (for the AL= -0.06 and AR=0.06) and δA_2 is (for the AL= -0.04 and AR=0.04). The average shift ratio could be defined by **Eq.(5.21)** is $\delta A_1 \approx 0$ and $\delta A_2 \approx 0$.
- Using **Eq.(5.10)**, the intermediate fuzzy consequent reference point is computed, where the reference point of the left triangular ($GPx.BL' = 5.96$) and the reference point of the right triangular ($GPx.BR' = 7.86$).
- Using **Eq.(5.23)** and the average of the shift ratios, we derive the reference points of the left triangular $GPx.BL^*= 5.96$ and the reference point of the right triangular $GPx.BR^*= 7.86$.
- According to **Eq.(5.12)** and **Eq.(5.13)**, we can compute the left and right fuzziness sides of the triangles required to find both extreme points of the fuzzy consequent set. The Left

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fuzziness side of the left triangular (*AL.PS*$_1$) and the Right fuzziness side of the right triangular (*AR.PS*$_3$) are ($AL.PS_1.B^* = 1.2$) and ($AR.PS_3.B^* = 2.0$), respectively.

- The fuzzy consequent result hence is formed as $B^* = (4.75\ 5.96\ 7.86\ 9.89)$.
- The conditions and fuzzy interpolative reasoning results are shown in **Table 9** and **Fig. 107** *(see Appendix A.8)*. There is no obvious indication for the HCL-FRI [42] and the HTY-FRI [53] to handle the fuzzy interpolation with multiple antecedent variables. From **Fig. 107** *(see Appendix A.8)*, we can see that the KH method [3], [25], [39], KHstabilized-FRI [5], VKK-FRI [4], the HS-FRI [22], CCL-FRI [52], MACI-FRI [6], IMUL-FRI [8], and CRF-FRI [7], and the Incircle-FRI all generated convex and normal results.

Table 9. *Fuzzy Interpolative Reasoning Results of Example MultiA1*

Attribute Values	Methods	Results of Fuzzy Interpolative reasoning
A_{11}=[0 4 5 6] A_{12}=[1 2 3 4] B_1=[0 2 3 4] A_{21}=[11 12 13 14] A_{22}=[12 14 15 16] B_2=[10 11 12 13] A^*_1=[6 7 9 11] A^*_2=[6 8 10 12]	KH-FRI [3],[25],[39] KHstabilized-FRI [5] VKK-FRI [4] CCL-FRI [52] HS-FRI [22] HTY-FRI [53] HCL-FRI [42] MACI-FRI [6] IMUL-FRI [8] CRF-FRI [7]	B^*=(5 6.15 8.0 9.88) B^*=(5 6.15 8.0 9.88) B^*=(4.9 6.11 8.12 9.88) B^*=(4.82 6.17 7.83 9.83) B^*=(4.37 5.55 7.48 9.33) B^*=(-) B^*=(-) B^*=(5.32 6.87 7.37 7.87 8.87) B^*=(4.3 6.6 7.6 10.68) B^*=(4.89 6.3 8.0 9.4)
The Incircle-FRI		**B^*=(4.75 5.96 7.86 9.89)**

Note: the sign (-) indicates no clear evidence for the method to handle the case in the example

Example MultiA2 [52], [57]:

- Incircle-FRI with multiple antecedent variables where the fuzzy rules are $A_{11} \wedge A_{12} \Rightarrow B_1$, $A_{21} \wedge A_{22} \Rightarrow B_2$, $A_{31} \wedge A_{32} \Rightarrow B_3$, and the observations A_1^* and A_2^* are given to determine the consequence B^*. All the rule antecedents, consequents, and the observation are trapezoidal fuzzy sets.
- Observation: $A^*_1 = [3.5\ 5\ 5\ 7]$, $A^*_2 = [5\ 6\ 6\ 7]$.
- According to **Eq.(5.20)**, we derive the intermediate observation of the first antecedent A'_1 is *(AL* A'_1 *=5.30 and AR* A'_1 *=5.74)* and for the second antecedent A'_2 is *(AL* A'_2 *=5.30 and AR* A'_2 *=5.74)*.
- Using the derived intermediate reference points (GP), the computed shift ratio from each antecedent domain are determined by **Eq.(5.22)**, where δA_1 = *(for the AL= -0.373 and AR= 0.0373)* and δA_2 = *(for the AL= -0.0373 and AR=0.0373)*. The average shift ratio based on **Eq.(5.21)** is $\delta A_1 \approx 0$ and $\delta A_2 \approx 0$.

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- Using **Eq.(5.10)**, the intermediate fuzzy consequent reference point is computed where *GPx.BL′ = 6.27* and *GPx.BR′ = 6.27*.
- Using **Eq.(5.23)** and the average of the shift ratios, we derive the reference points of the left triangular $GPx.BL^{*}= 6.27$ and the reference points of the right triangular $GPx.BR^{*}= 6.27$.
- According to **Eq.(5.12)** and **Eq.(5.13)**, we can compute the left and right fuzziness sides of the triangles that is required to find both extreme points of the fuzzy consequent set. The Left fuzziness side of the left triangular ($AL.PS_1$) and the Right fuzziness side of the right triangular ($AR.PS_3$) are ($AL.PS_1.B^* = 1.54$) and ($AR.PS_3.B^* = 1.27$), respectively.
- The fuzzy consequent result hence is formed as $B^* = (4.73\ 6.27\ 6.27\ 7.55)$.
- The fuzzy interpolative reasoning methods results are shown in **Table 10** and **Fig. 108** (*see Appendix A.9*
-), there is no obvious indication for the HCL-FRI [42] and the HTY-FRI [53] to handle the fuzzy interpolation with multiple antecedent variables. From **Fig. 108** (*see Appendix A.9*
-), we can see that the KH method [3], [25], [39], KHstabilized-FRI [5], VKK-FRI [4], the HS-FRI [22], CCL-FRI [52], MACI-FRI [6], IMUL-FRI [8], and CRF-FRI [7], and the Incircle-FRI all generated convex and normal results.

Table 10. *Fuzzy Interpolative Reasoning Results of Example MultiA2*

Attribute Values Folder 1	Methods	Results of Fuzzy Interpolative reasoning
A_{11}=[0 1 1 3]	KH-FRI [3],[25],[39]	B^*=(4.67 6.24 6.24 7.57)
A_{12}=[1 2 2 3]	KHstabilized-FRI [5]	B^*=(6.21 7.66 7.66 8.9)
B_1=[0 2 2 3]	VKK-FRI [4]	B^*=(4.87 6.26 6.26 7.53)
A_{21}=[8 9 9 10]	CCL-FRI [52]	B^*=(4.79 6.22 6.22 7.54)
A_{22}=[7 9 9 10]	HS-FRI [22]	B^*=(6.19 7.65 7.65 8.96)
B_2=[9 10 10 11]	HTY-FRI [53]	B^*=(-)
A_{31}=[11 13 13 14]	HCL-FRI [42]	B^*=(-)
A_{32}=[11 12 12 13]	MACI-FRI [6]	B^*=(5.23 6.57 6.57 7.57)
B_3=[12 13 13 14]	IMUL-FRI [8]	B^*=(4.1 6.25 6.25 8.89)
A^*_1=[3.5 5 5 7]	CRF-FRI [7]	B^*=(5.39 6.25 6.25 7.25)
A^*_2=[5 6 6 7]		
The Incircle-FRI		**B^*=(4.73 6.27 6.27 7.55)**

Note: the sign (-) indicates no clear evidence for the method to handle the case in the example

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5.2.3. Incircle-FRI with extrapolation

Example Ext1:

- The Incircle-FRI with multiple antecedent variables where the fuzzy rules are $A_1 \Rightarrow B_1$, $A_2 \Rightarrow B_2$, and the observation A^* is given to determine the consequence B^*. All the rule antecedents, consequents, and the observation are triangular fuzzy sets (with single antecedent).
- Observation: $A^* = [0\ 1\ 3]$.

Now, let us study the situation of the fuzzy extrapolation, where the problem exists when rules are chosen for interpolation, all rules appear on a single side of the observation. **Fig. 109** (see *Appendix A.10*

) illustrates an example with both fuzzy rules appearing on the right side of the observation.

- The (GpA') of the intermediate fuzzy set is calculated using **Eq.(5.20)** as $GpA' = 6.33$.
- By the derived (GpA') of the fuzzy set, we can get shift ratio by **Eq.(5.21)**, $\delta_A = -1.32$.
- By **Eq.(5.10)**, the intermediate consequent's (GPB') is computed, where $GPx.B' = 5.99$, and using **Eq.(5.23)**, the reference point (GP) of the shifted fuzzy consequent set $GPx.B^* = 5.99$.
- According to **Eq.(4.11)** , the left and right fuzziness sides are $PS_1(B^*) = 0.83$ and $PS_3(B^*) = 1.28$, respectively.
- Using **Eq.(4.12)**, we can derive the extrapolated fuzzy consequent result as $B^* = (5.16\ 5.99\ 7.28)$.
- The fuzzy interpolative reasoning methods results, which are shown in **Table 11** and **Fig. 109** (*see Appendix A.10*
-), there is no obvious indication for the KH method [3], [25], [39], KHstabilized-FRI [5], VKK-FRI [4], HCL-FRI [42], HTY-FRI [53], MACI-FRI [6], IMUL-FRI [8], and CRF-FRI [7] to handle the fuzzy interpolation with extrapolation. **Fig. 109** (*see Appendix A.10*
-) shown that the KHstabilized-FRI [5] and the Incircle-FRI are generating a conclusion with convex and normal results.

Table 11. *Fuzzy Interpolative Reasoning Results of Example Ext1*

Attribute Values	Methods	Results of Fuzzy Interpolative reasoning
A1=[3.5 5 7]; A2=[8 9 10] ; B1=[3 4 5]; B2=[9 10 11]; A_obs=[0 1 3];	KH-FRI [3], [25],[39]	B*=(-)
	KHstabilized-FRI [5]	B*=(4.82 6 6 7.18)
	VKK-FRI [4]	B*=(-)
	HTY-FRI [53]	B*=(-)
	HCL-FRI [42]	B*=(-)
	MACI-FRI [6]	B*=(-)
	IMUL-FRI [8]	B*=(-)
	CRF-FRI [7]	B*=(-)
The Incircle-FRI		**B*=(5.16 5.99 7.28)**

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Note: the sign (-) indicates no clear evidence for the method to handle the case in the example

Example Ext2:

- The Incircle-FRI with multiple antecedent variables where the fuzzy rules are $A_{11} \wedge A_{12} \Rightarrow B_1$, $A_{21} \wedge A_{22} \Rightarrow B_2$, and the observations A_1^* and A_2^* are given to determine the consequence B^*. All the rule antecedents, consequents, and the observation are triangular fuzzy sets (with multiple antecedents).
- Observation: $A^*_1 = [3.5\ 5\ 7]$, $A^*_2 = [5\ 6\ 7]$.
- Now, let us consider the situation of the fuzzy extrapolation, where the problem exists when rules are chosen for interpolation, all rules appear on a single side of the observation. **Fig. 110** (see *Appendix A.11*
-) describes an example with both fuzzy rules appearing on the right side of the observations.
- Using **Eq.(5.20)**, the reference point (GpA') of the intermediate fuzzy set is calculated as $A'_1 = 10.32$ and $A'_2 = 9.97$.
- Using **Eq.(5.21)** and the derived reference points of the intermediate fuzzy sets, we can get an average shift ratio, $\delta A = -1.32$.
- By **Eq.(5.10)**, the intermediate consequent's (GPB') is computed, where $GPx.B' = 10.99$, and using **Eq.(5.23)**, the reference point (GP) of the shifted fuzzy consequent set $GPx.B^* = 7.02$.
- According to **Eq.(4.11)**, the left and right triangle fuzziness sides are $PS_1(B^*) = 0.87$ and $PS_3(B^*) = 1.42$, respectively.
- Using **Eq.(4.12)**, we can derive the extrapolated fuzzy consequent result as $B^* = (6.15\ 7.02\ 8.44)$.
- The conditions and fuzzy interpolative reasoning results are shown in **Table 12** and **Fig. 110** (*see Appendix A.11*
-). There is no obvious indication for the KH method [3], [25], [39], KHstabilized-FRI [5], VKK-FRI [4], HCL-FRI [42], HTY-FRI [53], MACI-FRI [6], IMUL-FRI [8], and CRF-FRI [7] to handle the fuzzy interpolation with extrapolation. From **Fig. 110** (*see Appendix A.11*
-), we can see that the KHstabilized-FRI [5] and the Incircle-FRI are generating a conclusion with convex and normal results.

Table 12. *Fuzzy Interpolative Reasoning Results of Example Ext2*

Attribute Values	Methods	Results of Fuzzy Interpolative reasoning

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A_{11}=[8 9 10]	KH-FRI [3],[25],[39]	B^*=(-)
A_{12}=[7 9 10]	KHstabilized-FRI [5]	B^*=(10 11 11.94)
B_1=[9 10 11]	VKK-FRI [4]	B^*=(-)
A_{21}=[11 13 14]	HTY-FRI [53]	B^*=(-)
A_{22}=[11 12 13]	HCL-FRI [42]	B^*=(-)
B_2=[12 13 14]	MACI-FRI [6]	B^*=(-)
A^*_1=[3.5 5 7]	IMUL-FRI [8]	B^*=(-)
A^*_2=[5 6 7]	CRF-FRI [7]	B^*=(-)
The Incircle-FRI		**B^*=(6.15 7.02 8.44)**

Note: the sign (-) indicates no clear evidence for the method to handle the case in the example

SUMMARY

Chapter 5 has presented some extensions of the proposed Incircle-FRI method introduced in chapter 4. The extensions include the following features: handling hexagonal membership functions and handling multiple fuzzy rules having multiple antecedents. Further, enabling the capability of extrapolation, for this reason, the original Incircle-FRI was extended with a modified general weight calculation and shift process, where the weight between the observation and the neighboring rules will be derived according to overall distance instead of using extreme (closest) rules distance of the original method. This weight computation is crucial for the implementation of the extrapolation capability. The shifting process creates an intermediate fuzzy rule using existing neighboring rules. A shift ratio is computed between this intermediate rule and the observation to shift the intermediate fuzzy consequence with the same ratio that ensures a more reasonable interpolated and extrapolated consequence fuzzy result. All results of the extensions Incircle-FRI method (Hexagonal fuzzy set, multidimensional antecedents, extrapolation capability) produce CNF fuzzy conclusion.

Moreover, several experiments and comparisons were conducted, which were based on the extensions of the Incircle-FRI, aiming to prove the performance of the Incircle-FRI with different FRI methods. From the experimental results, we can see that the proposed method is considered one of the best current FRI methods. Consequently, the proposed method is a suitable approach to be implemented as an inference system.

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Theses related to Chapter 4 and Chapter 5:

Thesis. I:

I introduced a new method for the fuzzy rule interpolation concept called "Incircle-FRI", which is based on the Incircle of a triangular fuzzy number, the Gergonne Point as a "reference point" of the inside circle of triangular fuzzy set, and the fuzziness sides of the triangular. The Incircle-FRI conclusion is calculated by holding the same rate of the weights among the observation and the two rule antecedents, and the conclusion and the two corresponding rule consequents with the Gergonne Points (for the reference point of the conclusion), and with the "fuzziness sides" (for left and right fuzziness the shape of the conclusion). The "Incircle-FRI" is always generating a triangular CNF conclusion, if the antecedents and the consequents are triangular CNF sets, even if the fuzzy rule-base is sparse. I conclude that the proposed method is a suitable approach to be implemented as an inference system.

Thesis. II:

I introduced an extension of the "Incircle-FRI" to be able to handle trapezoidal and hexagonal fuzzy sets, which is by decomposing their membership function shapes into multiple triangulars, and multiple Incircle triangular fuzzy numbers with the Gergonne Points as reference points. I conclude that the extended "Incircle-FRI" can generate a trapezoidal, or hexagonal CNF conclusion if the antecedents and the consequents are all trapezoidal, or all hexagonal CNF sets, even if the fuzzy rule-base is sparse. Therefore, the Incircle-FRI method is a suitable approach to be implemented as an inference system with trapezoidal and hexagonal fuzzy sets.

Thesis. III:

I introduced an extension of the "Incircle-FRI" to be able to handle multiple fuzzy rules having multiple fuzzy antecedents. I used a modification weight estimate and included a shift technique to ensure to interpolate the consequent fuzzy result to be more logical and also to enable the capability for extrapolation. I conclude that the extensions of the "Incircle-FRI" always produce CNF conclusion, for all the handled antecedents and consequents configuration of the original method even if the fuzzy rule-base is sparse. Therefore, the Incircle-FRI method is a suitable approach to be implemented as an inference system with these extensions.

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The results introduced in chapter 4 and chapter 5 support the statement of Thesis I, II, III, and [99].

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INTRODUCTION

Many of the FRI methods suffer from not satisfying some FRI conditions (see chapter 2 subsection 2.6) related to the type of applicable linguistic terms and rule-base structure. A proper benchmark system could be built by analysing a set of conditions of the fuzzy sets such as (core, boundary, slopes, etc.) to compare the performance of different FRI conclusions. The construction of such a benchmark system is not straightforward because of numerous FRI methods and their special requirements. A good solution is to construct an FRI benchmark system based on the common criteria of the FRI methods according to fuzzy rule-base, fuzzy values, and observation configurations. Some of the required properties are not held in case of a given FRI method, e.g., the first method of the FRI "KH method". In this chapter, we will study all cases of the KH-FRI method to construct the initial FRI benchmark, highlighting the problematic situations for KH-FRI. Additionally, the benchmark will be used as a baseline to compare and evaluate the performance of existing and upcoming FRI methods.

6.1. A Survey Study Related to "Normality" and "Linearity" Properties to Compare FRI Methods Based on Arbitrary Examples

Before discussing the process of constructing a benchmark system for the CNF and PWL properties, I will present a survey study to explain the difference between the CNF and PWL properties. This survey study aims to use (arbitrary examples) to compare the results of the FRI methods according to the normality and linearity of the fuzzy conclusion using different features.

6.1.1. Arbitrary examples of experiments

We will present and discuss different examples (arbitrary examples) to compare the FRI methods (KH, KH Stabilized, MACI, IMUL, CRF, VKK, GM, FRIPOC, LESFRI, and SCALE MOVE) according to CNF and PWL properties. Various features are used for comparing: "the number of dimensions", "the shape of membership functions", and "the number of membership functions". The triangular, trapezoidal, and singleton membership functions are used to describe the antecedent, consequent, and observation. The FRI methods selected are tested by the FRI MATLAB toolbox.

The arbitrary examples will be described in details as follows:

- Examples ***FRI_EX1*** and ***FRI_EX2*** describe the antecedent and consequent by a single dimension, these examples will compare the results based on the difference between the number of the fuzzy sets by the same membership functions for antecedent, consequent, and observation.

- The Example ***FRI_EX3*** will represent the antecedent and consequent by a single dimension; the same number of the fuzzy sets is used for the antecedent, consequent part. This example describes the antecedent, consequent by a different membership function, where the observation represented by the trapezoidal membership function.
- Examples ***FRI_EX4*** and ***FRI_EX5*** were selected to show the results by the same membership functions of the antecedents and consequent using different shapes of the observation.
- Examples ***FRI_EX6*** and ***FRI_EX7*** describe by three dimensions of the antecedent parts and single dimension for the consequent part.

Table 13 summarizes the arbitrary examples. The antecedents and observations are shown in **Fig. 56 - Fig. 62**, the consequents part and conclusions have appeared in **Fig. 63 - Fig. 72**.

Table 13. *The Arbitrary Examples*

Examples	No. Dimensions		Type of Membership Functions			No. of Membership Functions	
	Antecedents	Consequents	Antecedents	Consequents	Observations	Antecedents	Consequents
Example FRI_EX1	1	1	**Tr**iangular	**Tr**iangular	**Tr**iangular	2	2
Example FRI_EX2	1	1	**Tr**iangular	**Tr**iangular	**Tr**iangular	4	4
Example FRI_EX3	1	1	**Tr**iangular	**Trap**ezoidal	**Trap**ezoidal	4	4
Example FRI_EX4	1	1	**Trap**ezoidal	**Trap**ezoidal	**Sing**leton	4	4
Example FRI_EX5	1	1	**Tr**iangular	**Tr**iangular	**Sing**leton	4	4
Example FRI_EX6	3	1	**Tr**iangular	**Tr**apezoidal	**Tr**iangular	3	3
Example FRI_EX7	3	1	**Tr**iangular	**Tr**apezoidal	**Sing**leton	3	3

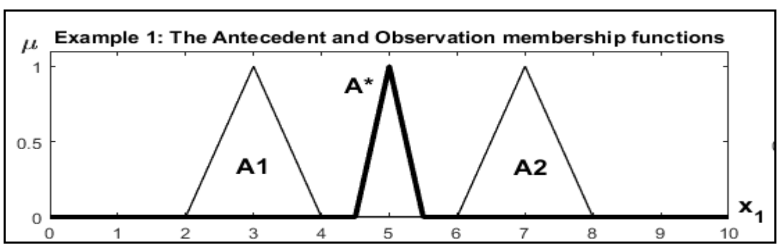

Fig. 56. *The Antecedent and Observation Fuzzy Sets Related to Example (FRI_EX1)*

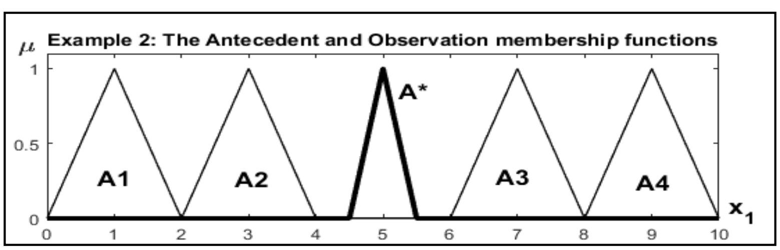

Fig. 57. *The Antecedent and Observation Fuzzy Sets Related to Example (FRI_EX2)*

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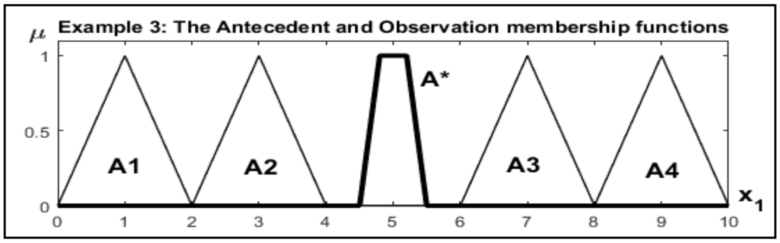

Fig. 58. *The Antecedent and Observation Fuzzy Sets Related to Example (FRI_EX3)*

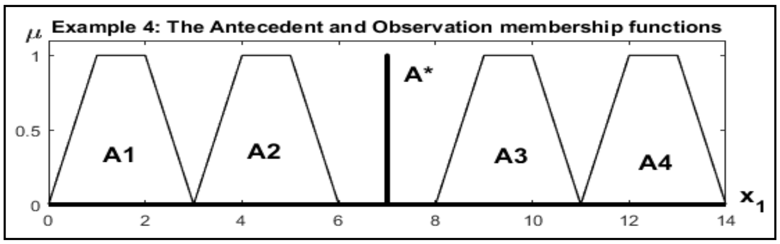

Fig. 59. *The Antecedent and Observation Fuzzy Sets Related to Example (FRI_EX4)*

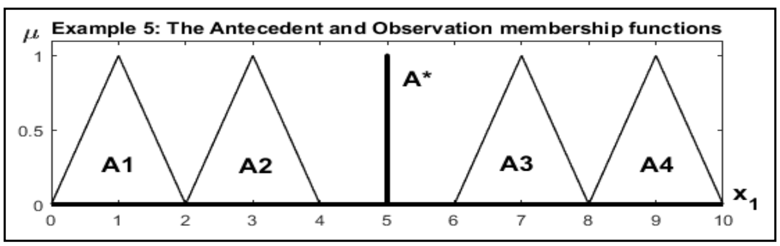

Fig. 60. *The Antecedent and Observation Fuzzy Sets Related to Example (FRI_EX5)*

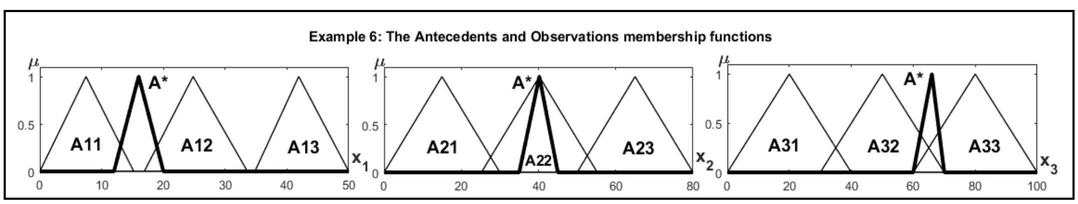

Fig. 61. *The Antecedents and Observations Fuzzy Sets Related to Example (FRI_EX6)*

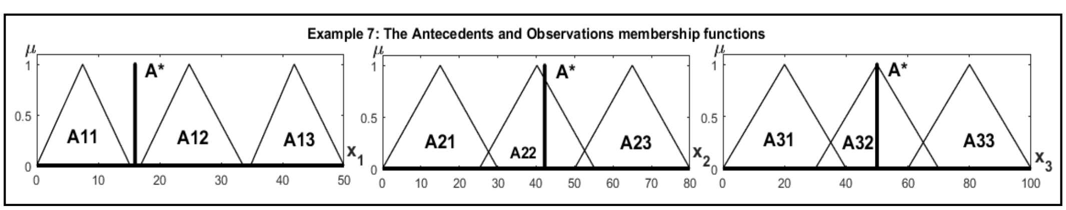

Fig. 62. *The Antecedents and Observations Fuzzy Sets Related to Example (FRI_EX7)*

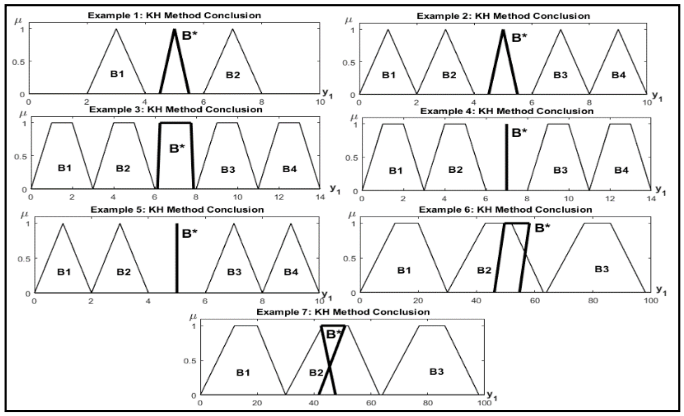

Fig. 63. *Arbitrary Examples: KH Conclusions*

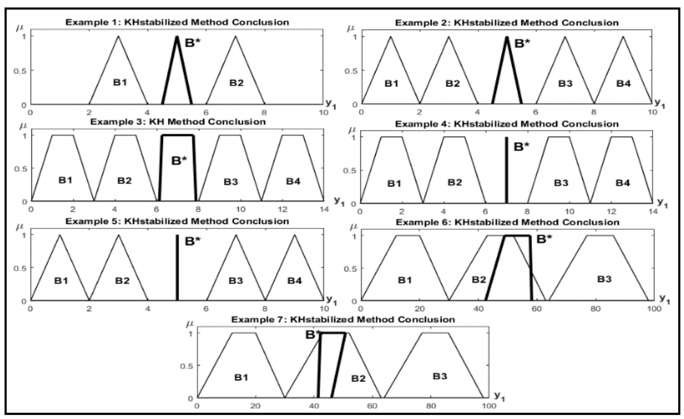

Fig. 64. *Arbitrary Examples: KH Stabilized Conclusions*

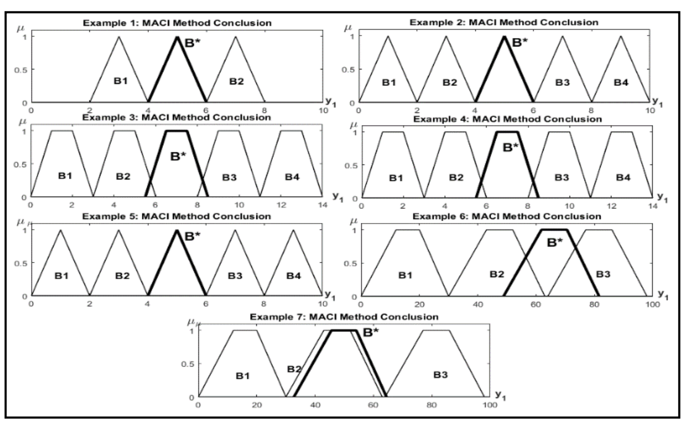

Fig. 65. *Arbitrary Examples: MACI Conclusions*

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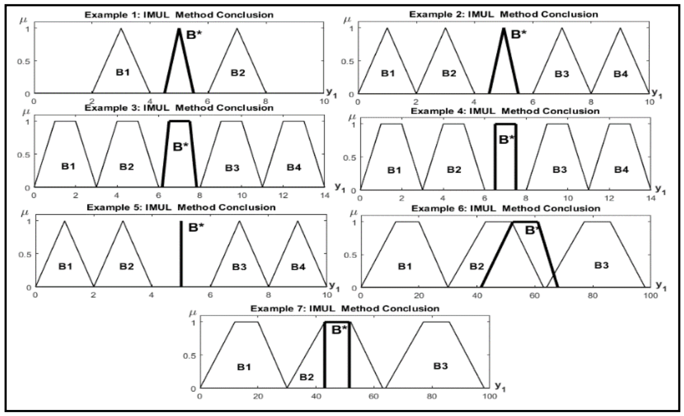

Fig. 66. *Arbitrary Examples: IMUL Conclusions*

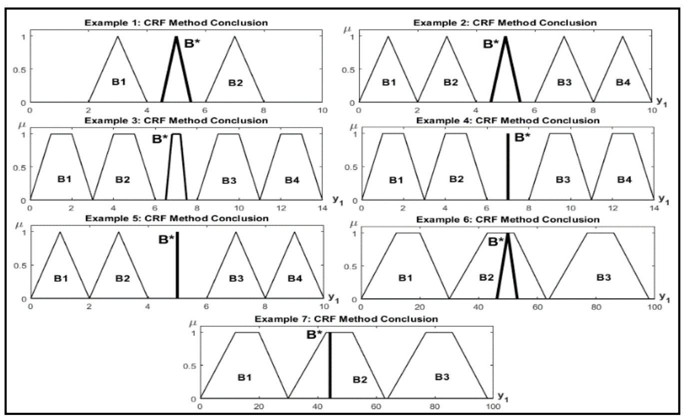

Fig. 67. *Arbitrary Examples: CRF Conclusions*

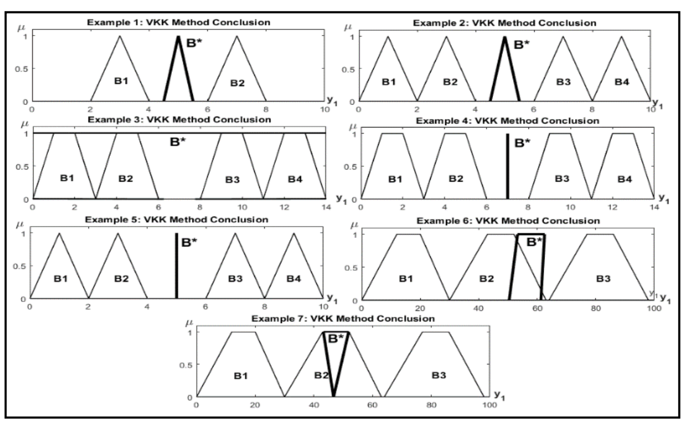

Fig. 68. *Arbitrary Examples: VKK Conclusions*

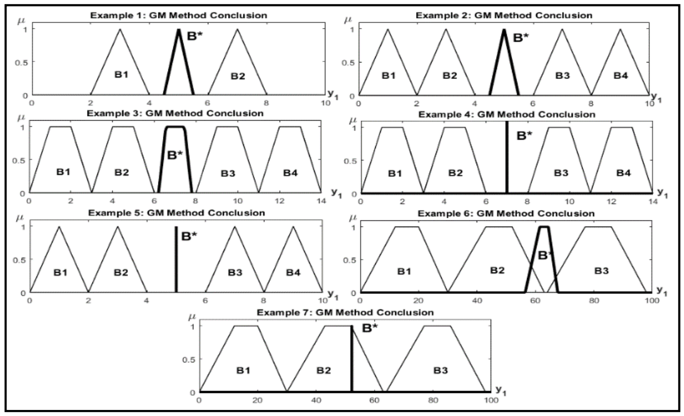

Fig. 69. *Arbitrary Examples: GM Conclusions*

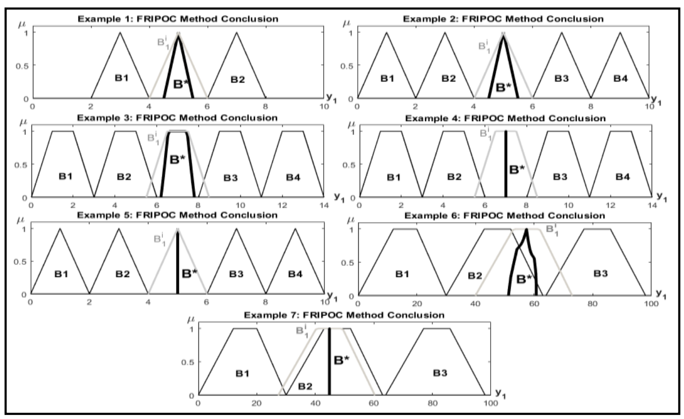

Fig. 70. *Arbitrary Examples: FRIPOC Conclusions*

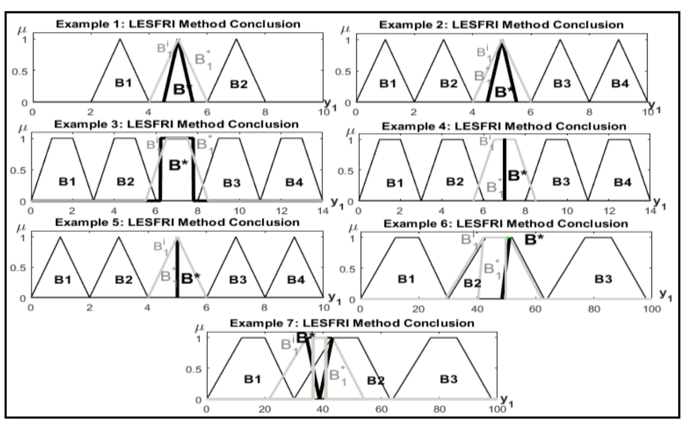

Fig. 71. *Arbitrary Examples: LESFRI Conclusions*

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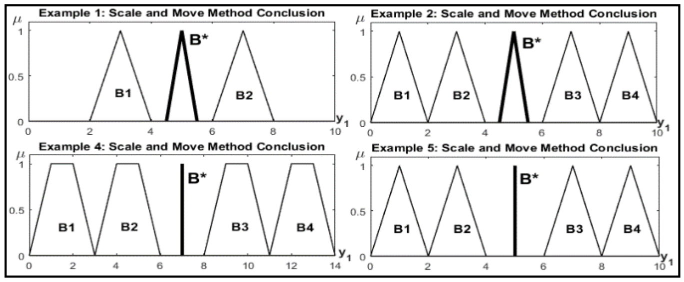

Fig. 72. Arbitrary Examples: Scale and Move Conclusions

6.1.2. Discussion of arbitrary examples results:

The results mentioned above of the arbitrary examples conclude the following:

- According to the antecedents and observations are shown in **Fig. 56, Fig. 57, Fig. 59,** and **Fig. 60**, the FRI methods (KH, KH Stabilized, MACI, IMUL, CRF, VKK, GM, FRIPOC, LESFRI, and SCALE MOVE) could be a suitable approach to be implemented as an inference system, in case a single dimension antecedent, where the antecedent and consequent have the same type of membership functions (*Triangular / Trapezoidal*), despite the type of observation membership function, **Fig. 63 - Fig. 72** describe the results of Examples (*FRI_EX1*, *FRI_EX2*, *FRI_EX4, and FRI_EX5*).
- Concerning the antecedents and observations are shown in **Fig. 58**, the FRI methods (KH, KH Stabilized) could be a suitable approach to be implemented as an inference system, in case a single dimension antecedent, in which the antecedent and consequent have a different type (*Triangular and Trapezoidal*), respectively, based on the type of the observation, **Fig. 63** and **Fig. 64** represented the results of the Example (*FRI_EX3*).
- According to the antecedents and observations are shown in **Fig. 58**, the FRI methods (MACI, IMUL, CRF, GM, FRIPOC, and LESFRI) could be a suitable approach to be implemented as an inference system, in case a single dimension, where the antecedent and consequent have a different type (*Triangular / Trapezoidal*), regardless of the type of the observation, **Fig. 65 - Fig. 67** and **Fig. 69 - Fig. 71** illustrated the results of the Example (*FRI_EX3*).
- Regarding the antecedents and observations are shown in **Fig. 61** and **Fig. 62**, the FRI methods (MACI and GM) could be a suitable approach to be implemented as an inference system, in case multi-dimension antecedents, where the antecedent and consequent have a different type of membership functions (*Triangular / Trapezoidal*), despite the type of observation membership function, **Fig. 65** and **Fig. 69** described the results of Examples (*FRI_EX6 and FRI_EX7*).
- According to the antecedents and observations are shown in **Fig. 61** and **Fig. 62**, the FRI methods (IMUL and CRF) could be a suitable approach to be implemented as an inference system, in case multi-dimension antecedents, where the antecedent and consequent have a different type of membership functions (*Triangular and Trapezoidal*), regardless of the type

of observation membership function, **Fig. 66** and **Fig. 67** described the results of Examples (*FRI_EX6 and FRI_EX7*).

- Regarding the antecedents and observations are shown in **Fig. 62**, the FRI method (FRIPOC) could be a suitable approach to be implemented as an inference system, in case multi-dimension antecedents, where the antecedent and consequent have a different type of membership functions (*Triangular and Trapezoidal*), in case of the type of observation membership function is singleton, **Fig. 70** shown the results of the Example (*FRI_EX7*).
- On the other hand, regarding the antecedents and observations are shown in **Fig. 58**, the FRI method (VKK) suffers from the abnormality in case a single dimension antecedent, where the antecedent and consequent have a different type (*Triangular and Trapezoidal*), respectively, based on the type of the observation, **Fig. 68** illustrated the results of the Example (*FRI_EX3*).
- Referring to the antecedents and observations shown in **Fig. 61** and **Fig. 62**, the FRI methods (KH, KH Stabilized, and VKK) suffer from the abnormality in case multi-dimension antecedents. In contrast, the antecedent and consequent have a different type of membership functions (*Triangular / Trapezoidal*), regardless of the type of observation membership functions, **Fig. 63, Fig. 64** and **Fig. 68** described the results of Examples (*FRI_EX6 and FRI_EX7*).
- According to the antecedents and observations are shown in **Fig. 61**, the FRI method (FRIPOC) suffers from the piecewise linearity in multi-dimension antecedents. In contrast, the antecedent and consequent have a different type of membership functions (*Triangular / Trapezoidal*), in case the type of observation membership function is triangular, which the results shown in **Fig. 70** and described in the Example (*FRI_EX6*).
- Regarding the antecedents and observations are shown in **Fig. 61** and **Fig. 62**, the FRI method (LESFRI) suffers from the abnormality, in case multi-dimension antecedents. In contrast, the antecedent and consequent have a different type of membership functions (*Triangular and Trapezoidal*), in case the type of observation membership function is triangular, which the results shown in **Fig. 71** and defined in Examples (*FRI_EX6 and FRI_EX7*).

Based on the above, we are going to construct the initial benchmark examples for two of the most important recommended properties of the FRI concept, which are CNF and PWL properties, to be a baseline for comparing and evaluating FRI methods. The survey study in (6.1) gave a general description for comparing FRI methods according to normality and linearity based on different features, but these examples are not sufficient to be used as benchmark examples for comparing FRI methods with CNF and PWL properties. Therefore, in the following subsections, the initial benchmark examples of CNF and PWL properties will be constructed, the properties are following the KH-FRI method, taking into consideration

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investigating all the conditions of the core, boundary, and slopes of the fuzzy rule bases and observation, I will present in details these initial benchmark examples below.

6.2. FRI Benchmark Examples of the CNF Property for the Koczy-Hirota Interpolation Method.

The original KH-FRI produces the output based on α-cuts. The most significant benefit of the KH-FRI is its low computational complexity. Despite many advantages, in some antecedent fuzzy set configuration, the KH-FRI suffers from the abnormality of the conclusion (see more details in [17], [32]). The study in [18], [41] discusses the normality property and gives some boundary conditions for the observation, the antecedent, and consequent fuzzy sets. Where the normality of the conclusion necessarily holds.

The main goal of this subsubsection is to take these boundary conditions and construct *CNF_Benchmark Examples* to highlight the problematic properties of the original KH Fuzzy Rule Interpolation. Besides, this *CNF_Benchmark Examples* could be used for testing other FRI methods against these ill conditions. All *CNF_Benchmark Examples* introduced in this subsubsection are implemented by the MATLAB FRI Toolbox [24], [26], which provides an easy-to-use framework for FRI applications.

6.2.1. Preliminaries and basic definitions related to CNF property

A fuzzy set defined on a universe of discourse that holds total ordering is a convex and normal fuzzy (CNF) set. If it has a height equal to one, and having membership grade of any elements between two other elements greater than, or equal to the minimum membership degree of these tow boundary elements. I.e a convex fuzzy set can be defined by $(\forall x, y \in U), (\forall \lambda \in [0,1]): (\mu_A(\lambda x + (1-\lambda y)) \geq min\ (\mu_A(x), \mu_A(y))))$.

Fig. 73 describes some properties of the membership functions. The support of the fuzzy set is the set of all elements in the universe of discourse with a greater than zero membership degree. The α-cut and the strong α-cut of a fuzzy set is the crisp subset of the universe where the membership degrees are higher (*strong α-cut*), or higher, or equal (*α-cut*) than a specified α value. The kernel of a fuzzy set is the crisp subset of the universe, where the membership degrees are equal to *1*. The width of a convex fuzzy set is the length of the support, an interval in the case of a convex fuzzy set. (see subsubsection 4.1.1) for more details about these properties)

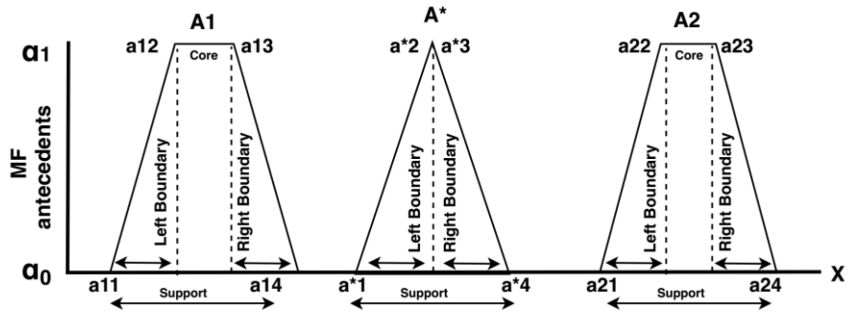

Fig. 73. Support and α-Cuts of Triangular and Trapezoidal Fuzzy Sets

A typical example of a sparse fuzzy rule-base was presented in the KH-FRI reasoning method in [42]. It could be briefly described as follows. From the rule-base, the two closest surrounding fuzzy rules to the observation are taken into consideration only (see **Fig. 73,** the observation and the two surrounding antecedent fuzzy sets):

If X is A_1 then Y is B_1
If X is A_2 then Y is B_2

The two rules will be abbreviated as $A_1 \Rightarrow B_1$ and $A_2 \Rightarrow B_2$, respectively. Suppose that these two rules are adjacent, as shown in **Fig. 74**. Thus, we can see that when observation A^* has no overlapping with fuzzy sets A_1 or A_2, none of the rules are firing, no results could be obtained by classical fuzzy reasoning.

If X is A_1 then Y is B_1
If X is A_2 then Y is B_2
Observation: X is A^*

Conclusion: $Y = (B^*)$

FRI reasoning methods could provide interpolated conclusion B^* when the observation A^* does not overlap with any of the rule antecedents A_1 and A_2. According to the interpolation concept that was suggested by Koczy and Hirota in [3], [25], and [14]. In the following, some definitions related to linear interpolation (KH-FRI) can be introduced:

• **Definition of the Preceding CNF Sets:** Referring to all fuzzy sets that must be normal and convex in the universe X_i by $P(X_i)$. The α-cuts are intervals. Then for $(A_1, A_1 \in P(X_i))$, *if* $(\forall\alpha \in (0,1])$, A_1 is precedes A_2 $(A_1 < A_2)$ if:

$$inf(A_{1\alpha}) < inf(A_{2\alpha}),\ sup(A_{1\alpha}) < sup(A_{2\alpha}) \quad (6.1)$$

Where $A_{1\alpha}$ and $A_{2\alpha}$ are α-cut sets of A_1 and A_2, respectively, $inf\,(A_{i\alpha})$ is the infimum of $A_{i\alpha}$ and $sup\,(A_{i\alpha})$ is the supremum of $A_{i\alpha}$ $(i= 1,2)$.

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• **Definition of the Lower and Upper Fuzzy Distances:** Given a fuzzy relation $(R\prec: (A_1, A_2) \mid A_1, A_2 \in P(X)), (A_1 \prec A_2)$, if fuzzy sets A_1 and A_2 satisfy $R\prec$, the lower (dL) and the upper (dU) fuzzy distances between A_1 and A_2 by the resolution principles [46], [47], it can be defined by **Eq.(6.2)**:

$$\begin{gathered} dL\,(A_1, A_2): R\prec \rightarrow P([0,1]) \\ \mu dL(\delta): \textstyle\sum_{\alpha \in [0,1]} \alpha \,/ d(inf(A_{1\alpha}), inf(A_{2\alpha})) \\ dU\,(A_1, A_2): R\prec \rightarrow P([0,1]) \\ \mu dU(\delta): \textstyle\sum_{\alpha \in [01]} \alpha \,/ d(sup(A_{1\alpha}), sup(A_{2\alpha})) \end{gathered} \tag{6.2}$$

Where $\delta \in [0,1]$ and d is the Euclidean distance, or more generally the Minkowski distance.

• **Definition of the KH Linear Fuzzy Rule Interpolation:** When $A_1 \Rightarrow B_1$, $A_2 \Rightarrow B_2$ be disjoint fuzzy rules on the universe of discourse $X \times Y$, and A_1, A_2, B_1, and B_2 be fuzzy sets on X and Y, respectively. Assume that A^* is the observation of the input universe X. *If* $(A_1 < A^* < A_2)$ then the KH linear fuzzy rule interpolation between R_1 and R_2 is defined by **Eq.(6.3)**:

$$d(A_1, A^*) : d(A^*, A_2) = d(B_1, B^*) : d(B^*, B_2) \tag{6.3}$$

Where d refers to the fuzzy distance according to *Definition the KH Linear Fuzzy Rule Interpolation* that could be used between the fuzzy sets (A_1, A^*, A_2) and (B_1, B_2).

• **Definition of the Lower and Upper Distances Between α-cuts:** Let A_1 and A_2 be fuzzy sets on the universe of discourse X with $|X| < \infty$, then the lower and upper distances between α-cuts sets $A_{1\alpha}$ and $A_{2\alpha}$ are determined via **Eq.(6.4)**:

$$\begin{gathered} dL\,(A_{1\alpha}, A_{2\alpha}) = d(inf(A_{1\alpha}), inf\,(A_{2\alpha})), \\ dU\,(A_{1\alpha}, A_{2\alpha}) = d(sup(A_{1\alpha}), sup\,(A_{2\alpha})) \end{gathered} \tag{6.4}$$

According to *Definitions of the Preceding CNF Sets, the KH Linear Fuzzy Rule Interpolation and the Lower and Upper Distances Between α-cuts* the FERI of (dU) and (dL) α-cuts, the formula can be rewritten as **Eq.(6.5)**:

$$\begin{gathered} dL\,(A^*, A_{1\alpha}): dL\,(A^*, A_{2\alpha}) = dL\,(B^*, B_{1\alpha}): dL\,(B^*, B_{2\alpha}) \\ dU\,(A^*, A_{1\alpha}) : dU\,(A^*, A_{2\alpha}) = dU\,(B^*, B_{1\alpha}) : dU\,(B^*, B_{2\alpha}) \end{gathered} \tag{6.5}$$

Thus, the infimum (*inf*) and supremum (*sup*) of the conclusion can be determined by **Eq.(6.6)** and **Eq.(6.7)**:

$$\text{Inf}(B^*_\alpha) = \frac{dL(A^*_\alpha, A_{1\alpha}) \times \inf(B_{2\alpha}) + dL(A^*_\alpha, A_{2\alpha}) \times \inf(B_{1\alpha})}{dL(A^*_\alpha, A_{1\alpha}) + dL(A^*_\alpha, A_{2\alpha})}$$

$$\text{Sup}(B^*_\alpha) = \frac{dU(A^*_\alpha, A_{1\alpha}) \times \sup(B_{2\alpha}) + dU(A^*_\alpha, A_{2\alpha}) \times \sup(B_{1\alpha})}{dU(A^*_\alpha, A_{1\alpha}) + dU(A^*_\alpha, A_{2\alpha})} \quad (6.6)$$

then,

$$B^*_\alpha = (\inf(B^*_\alpha); \sup(B^*_\alpha)). \quad (6.7)$$

Finally, consequence B^* can be constructed by **Eq.(6.8)**:

$$B^* = \bigcup_{\alpha \in [0,1]} \alpha . B^*_\alpha \quad (6.8)$$

6.2.2. The original KH fuzzy linear interpolation

The original Koczy and Hirota interpolation (later referred as KH-FRI) [3], [7], [16], [25], [39], [40] requires the antecedents and consequences fuzzy sets to be convex and normal (CNF) [15], [43]. In this case, the approximated conclusion can be generated by decomposing the fuzzy sets into α-cuts. The KH-FRI is defined for a single-dimensional antecedent space, for two rules, whose antecedents surround the observation:

$$(A_1 \prec A^* \prec A_2)$$
$$And$$
$$(B_1 \prec B_2)$$

According to the concept of fuzzy distance [16] in KH-FRI (*see Definition of the Preceding CNF Sets*), the fuzzy distance of two CNF sets can be defined by the distance of lower and upper endpoints of their α-cuts. The "linear interpolation" idea of the KH-FRI is that the rate of the upper and lower fuzzy distances between observation and antecedents. It must be the same as the rate of the fuzzy distances between the two rule conclusions and the consequent. Therefore, regarding the previous definitions and resolution principles of fuzzy sets, the conclusion B^* for the KH-FRI method is produced directly based on α-cuts of the observation and the two surrounding fuzzy rules.

A key advantage of the original KH approach is its low computational complexity for fuzzy rules. Since it deals with two rules only from the rule base during the determination of consequent. The antecedents of those rules are the closest flanking to the observation, $(A_1 \prec A^* \prec A_2)$ (See **Fig. 74**).

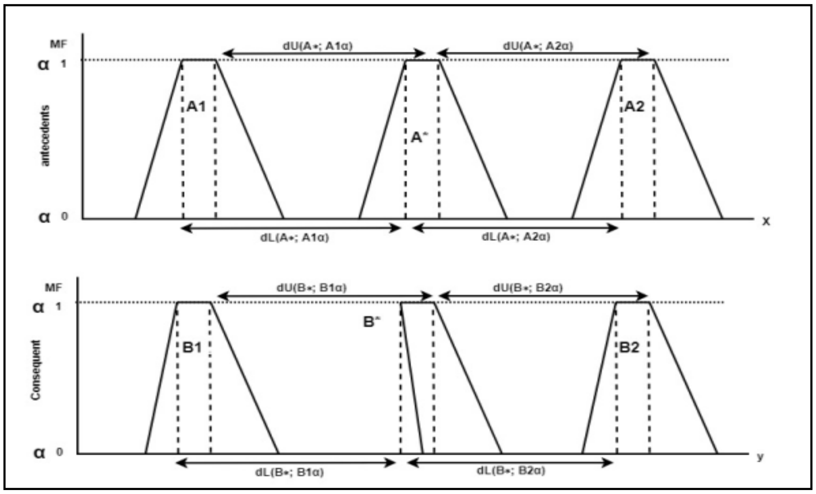

Fig. 74. *Fuzzy Interpolative Reasoning with an Invalid Conclusion for the KH-FRI*

On the other hand, searching for these two rules could be a computationally demanding task. Despite the advantages, in some rule and observation configurations, the conclusion can be abnormal, or not always directly interpretable. Therefore, in the following, the conditions of normal and abnormal conclusions will be presented in detail.

6.2.3. The convexity and normality of the KH-FRI conclusion

Several FRI techniques followed the resolution principle, which requires turning the problem of fuzzy interpolation into an infinite family of crisp interpolations. That is according to the α-cuts of the fuzzy rules and observation, then merging the results and conclude the fuzzy solution, (see **Eq.(6.8)**).

In the case of the KH-FRI, several necessary conditions must be held, requiring all fuzzy sets to be convex and normal (CNF). This condition guarantees that all α-cuts are intervals and exist. The CNF property of the conclusion fuzzy set can be checked if all α-cuts are connected. The KH-FRI cannot produce any results if the *α-cuts* are not connected (see cases in [35]). The conclusion is created as intervals by determining their lowest and highest endpoints. Therefore, the convexity condition is automatically satisfied. **Fig. 75** represents a convex and a non-convex fuzzy set.

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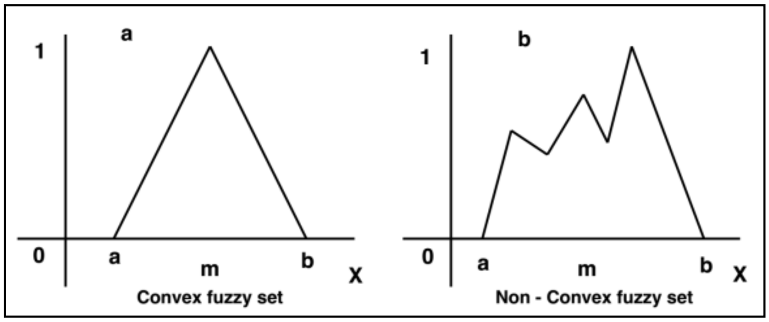

Fig. 75. *A Convex (a) and a Non-Convex (b) Fuzzy Set*

In contrast, the normality of the conclusion is not always satisfied. The conclusion is normal if the membership function assumes all values between *0* and *1*. So, the condition will be satisfied if $(inf(B^*_\alpha) \leq sup(B^*_\alpha))$ for all α. Otherwise, if the condition is not satisfied, the membership function will suffer from an abnormality, as shown in **Fig. 76**.

To generate the initial *CNF_Benchmark Examples*, we should collect which equations in [18], [41] have been used to determine and verify the normality of the conclusion. If the shape of the antecedent and consequent fuzzy sets is restricted to triangular and trapezoidal, the membership functions can be described by three or four points. In the case of trapezoidal, it has four values (a_1, a_2, a_3, a_4), and in case of triangular, it could consider as a special trapezoidal $a_2 = a_3$ (see **Fig. 73**).

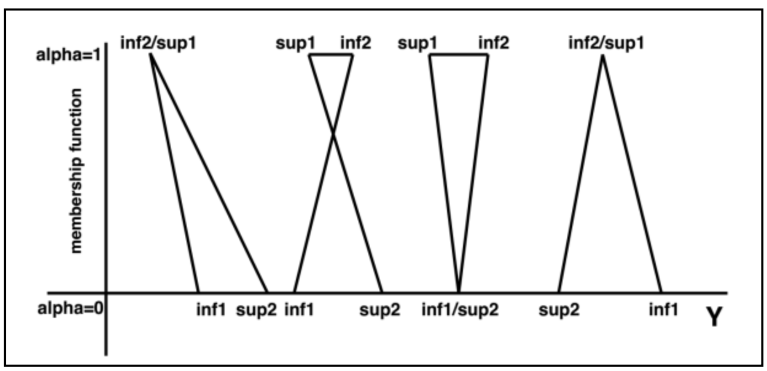

Fig. 76. *Forms of the Abnormal Conclusions*

Additionally, a singleton membership function is also a particular trapezoidal membership function, where all the values (a_1, a_2, a_3, a_4) are the same. Accordingly, the characteristic points of the KH-FRI conclusion can be defined by the following equations in [41]. The conclusion is that B^* is normal if and only if $(y_{inf1}, y_{inf2}, y_{sup1}, \text{ and } y_{sup2})$ are met the following conditions:

$$\begin{aligned} y_{inf1} &= \frac{(a_{21}-a_1^*)*b_{11}+(a_1^*-a_{11})*b_{21}}{a_{21}-a_{11}} \leq \\ y_{inf2} &= \frac{(a_{22}-a_2^*)*b_{12}+(a_2^*-a_{12})*b_{22}}{a_{22}-a_{12}} \leq \\ y_{sup1} &= \frac{(a_{23}-a_3^*)*b_{13}+(a_3^*-a_{13})*b_{23}}{a_{23}-a_{13}} \leq \\ y_{sup2} &= \frac{(a_{24}-a_4^*)*b_4+(a_4^*-a_{14})*b_{24}}{a_{24}-a_{14}} \end{aligned} \tag{6.9}$$

According to **Eq.(6.10)** - **Eq.(6.12)**, the conclusion core and boundary lengths can be determined. To verify the normality of the LefT Boundary (*LTB*) length of the conclusion, **Eq.(6.10)** could be applied:

$$Length.LT\,Bound1 \leq Length.LT\,Bound2 \tag{6.10}$$

where

$$Length.LT\,Bound1 = db_{LT\,B} \times (((Ka_{1LTB}+da_{1LTB}) \times (Ka_{2LTB}+da_{2LTB})) - ((Ka^*_{LTB}+da_{1LTB}) \times (Ka^*_{LTB}+da_{2LTB})))$$

$$Length.LT\,Bound2 = ((Ka_{1LTB}+da_{1LTB}) \times (da_{1LTB}+Ka^*_{LTB}) \times Kb_{2LT\,B}) + ((Ka_{2LTB}+da_{2LTB}) \times (da_{2LTB}+Ka^*_{LTB}) \times Kb_{1LTB})$$

The core length of the conclusion can be determined by **Eq.(6.11)** as follows:

$$Length.Core1 \leq Length.Core2 \tag{6.11}$$

where

$$Length.Core1 = db_{core} \times (((Ka_{1core}+da_{1core}) \times (Ka_{2core}+da_{2core})) - ((Ka^*_{core}+da_{1core}) \times (Ka^*_{core}+da_{2core})))$$

$$Length.Core2 = ((Ka_{1core}+da_{1core}) \times (da_{1core}+Ka^*_{core}) \times Kb_{2core}) + ((Ka_{2core} + da_{2core}) \times (da_{2core} + Ka^*_{core}) \times Kb_{1core})$$

For RighT Boundary (*RTB*) length of the conclusion can be determined by the following **Eq.(6.12)**:

$$Length.RT\,Bound1 \leq Length.RT\,Bound2 \tag{6.12}$$

where

$$Length.RT\,Bound1 = db_{RTB} \times (((Ka_{1RTB}+da_{1RTB}) \times (Ka_{2RTB}+da_{2RTB})) - ((Ka^*_{RTB}+da_{1RT}B) \times (Ka^*_{RTB}+da_{2RTB})))$$

$$Length.RT\,Bound2 = ((Ka_{1RTB}+da_{1RTB}) \times (da_{1RTB}+Ka^*_{RTB}) \times Kb_{2RT\,B}) +$$

$$((Ka_{2RTB}+da_{2RTB}) \times (da_{2RTB}+Ka^{*}_{RTB}) \times Kb_{1RTB})$$

where the parameters of the core length for **Eq.(6.11)** can be defined as follows:

$$Ka_{1core}=a_{13}-a_{12},\ Ka_{2core}=a_{23}-a_{22}$$
$$Kb_{1core}=b_{13}-b_{12},\ Kb_{2core}=b_{23}-b_{22}$$
$$Ka^{*}_{core}=x_3-x_2,\ da_{1core}=x_2-a_{13}$$

The *LTB* and *RTB* boundary can be constructed similarly to the core length parameters (as above parameters). From another point of view, the length ratio of the distance between the fuzzy sets of the antecedent with observation (Ka_i, Ka^*) and consequent (Kb_i) **Eq.(6.13)**, **Eq.(6.14)**, and **Eq.(6.15)** could also be used to check the normality (validity) of the conclusion, which can be defined as follows:

For the length ratio of the left Boundary:

$$RatioLT1= LTBound\ (K_{b1,b2}) / LTBound\ (K_{a1,a2}).$$
$$RatioLT2= LTBound(K_{a1,a2}) / (LTBound\ (Ka^{*}Ka_1)+ LTBound(Ka_2Ka^{*})). \qquad (6.13)$$

where

$$LTBound(K_{b1,b2})= b_{21} - b_{12},$$
$$LTBound(K_{a1,a2})= a_{21} - a_{12},$$
$$LTBound(Ka^{*}Ka_1)= a^{*}_1 - a_{12},$$
$$LTBound(Ka_2Ka^{*})= a_{21} - a^{*}_2.$$

For the length ratio of the core:

$$RatioC1= Core(K_{b1,b2})/Core(K_{a1,a2}),$$
$$RatioC2= Core(K_{a1,a2})/(Core(Ka^{*}Ka_1)+Core(Ka_2Ka^{*})). \qquad (6.14)$$

where

$$Core(K_{b1,b2})= b_{22}- b_{13},$$
$$Core(K_{a1,a2})= a_{22}- a_{13},$$
$$Core(Ka^{*}Ka_1)= a^{*}_2 - a_{13},$$
$$Core(Ka_2Ka^{*})= a_{22} - a^{*}_3.$$

For the length ratio of the right Boundary:

$$RatioRT1= RTBound\ (K_{b1,b2}) / RTBound\ (K_{a1,a2}).$$
$$RatioLT2= RTBound(K_{a1,a2}) / (RTBound\ (Ka{*}Ka1)+ RTBound(Ka_2Ka^{*})). \qquad (6.15)$$

where

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$$RTBound(K_{b1,b2}) = b_{23} - b_{14},$$
$$RTBound(K_{a1,a2}) = a_{23} - a_{14},$$
$$RTBound(Ka^{*}Ka_{1}) = a^{*}_{3} - a_{14},$$
$$RTBound(Ka_{2}Ka^{*}) = a_{23} - a^{*}_{4}.$$

6.2.4. Reference values the CNF property

According to the main corollaries in [18], [41], the normality of the KH-FRI conclusion can be determined as follows:

1) Corollary *CNFCR1*: $Ka_i = Kb_i = Ka^*$

If rules $A_1 \Rightarrow B_1$, $A_2 \Rightarrow B_2$, and the observation A^* have the same core and left-right boundary lengths as the antecedent (Ka_i) and consequent (Kb_i) fuzzy sets, the conclusion will always be normal. For this corollary, **Eq.(6.10) - Eq.(6.15)** could validate the normality.

2) Corollary *CNFCR2*: $Ka_i = KA$, $Kb_i = KB$

If the fuzzy set of the antecedent ($Ka_i = KA$) and the consequent ($Kb_i = KB$) have a uniform core and boundary lengths, then the conclusion fuzzy set is always normal if and only if the following conditions by **Eq.(6.16)** and **Eq.(6.17)** must hold:

For the core length:

- *If* $Ka^* \neq 0$

$$Length.Core1 \leq Length.Core2 \tag{6.16}$$

where

$$Length.Core1 = db_{core} \times (Ka_{core} - Ka^{*}_{core}),$$
$$Length.Core2 = Kb \times (da_{1core} + da_{2core} + 2 \times Ka^{*}_{core})$$

- *If* $Ka^* = 0$

$$Length.Core1 \leq Length.Core2 \tag{6.17}$$

where

$$Length.Core1 = db_{core} \times (Ka_{core} - Ka^{*}_{core}),$$
$$Length.Core2 = Kb_{core} \times da_{core}$$
and
$$da_{core} = a_{22} - a_{13}$$

For left and right boundary lengths, similar equations to the core length could be constructed.

3) Corollary *CNFCR3*: $Ka_i = Ka^*$, $Kb_i = KB$

In this corollary, if the antecedent fuzzy sets and observation have the same core and boundary lengths, and the fuzzy sets of the consequent have the same length too, then the

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conclusion fuzzy set is always normal. To verify the normality condition **Eq.(6.16)** and **Eq.(6.17)** for the core and boundary lengths are used.

4) Corollary *CNFCR4*: *The antecedents and consequences have uniform core length*

The conclusion fuzzy set is always normal if the length ratio of the distance between the fuzzy sets of the antecedent and consequent (*distance KB*) / (*distance KA*) does not exceed the length ratio of themselves. **Eq.(6.13)**, **Eq.(6.14)**, and **Eq.(6.15)** can be used to verify the normality condition, in other words, the consequents have not shorter length, i.e., the consequents are not less than fuzzy the antecedents.

Fig. 77 illustrates all the core and boundary CNF_Notations (*Ka*, *Kb*) that are used in **Eq.(6.10) - Eq.(6.15)**, as follows:

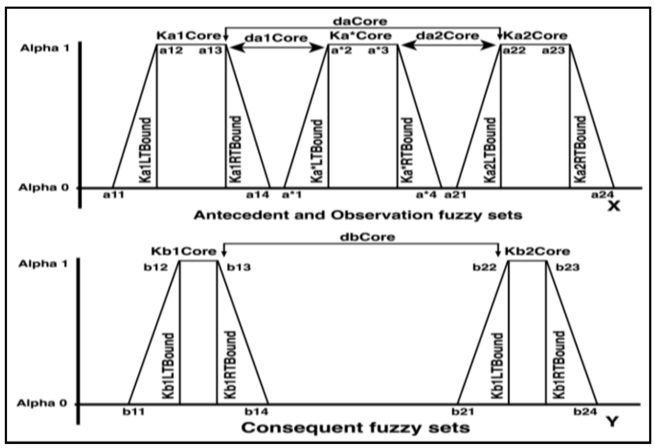


Fig. 77. *CNF_Notations Related to Core and Boundary Lengths of Trapezoidal Fuzzy Sets*

6.2.5. The KH-FRI CNF benchmark

In the following, the *CNF_Benchmark Examples* will be constructed to highlight the conditions of the normality conclusion of the KH-FRI. Various corollaries introduced to check the normality of the conclusion based on the core and (*Left-Right*) boundary lengths have a primary role in determining the normality. According to the prerequisites of the KH-FRI, one-dimensional antecedents and consequents with trapezoidal, triangular, and singleton fuzzy sets, and two rules of the rule-bases could be considered. In the rest of the subsubsection, all the calculations and figures were prepared by the fuzzy rule interpolation (FRI) toolbox. The current version of the FRI toolbox is freely available to download in [24].

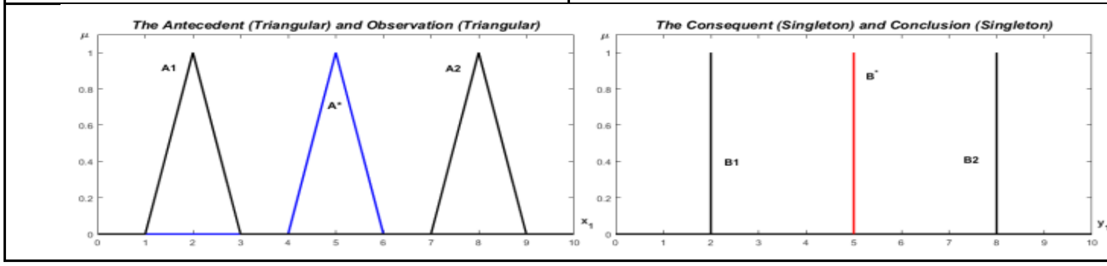
We will discuss the special cases where the conclusion of the KH-FRI is normal and abnormal according to the equations and corollaries explained previously.

First of all, the normality condition is always satisfied with the KH-FRI if any of the following cases are met:

- **Case *CNF.C1*:** When the core and boundary lengths of the observation are greater or equal than the antecedent fuzzy sets *(KA^* >= KA)*, if *(Ka_i = KA)*, the normality of the KH-FRI conclusion fuzzy set will always be satisfied. In this case, there is no restriction on the shape and size of the consequent (*KB*). **Table 14** illustrates the Example "*CNF.KH_NOR.C1*" that demonstrates ***Case CNF.C1***.

Table 14. *The Normal Conclusion of the KH-FRI with Fuzzy Sets According to Case CNF.C1*

Example ("*CNF.KH_NOR.C1*")	
CNF_Notations that prove case *CNF.C1* (KA^* >= KA), if (Ka_i = KA)	
The values of the fuzzy sets: A_1= [1 2 2 3] A_2= [7 8 8 9] B_1= [2 2 2 2] B_2= [8 8 8 8] A^*= [4 5 5 6] B^*= [5 5 5 5]	**The case of the Core and (LF and RF) Boundary conclusion:** The length (LFBound) is (*NORMAL*) The length (Core) is (*NORMAL*) The length (RFBound) is (*NORMAL*)
The length ratio between KA, Ka*, and KB: **LTBound**: *RatioLT1= 1.20, RatioLT2= 1.25* **Core**: *RatioC1 = 1, RatioC2= 1* **RTBound**: *RatioRT1= 1.20, RatioRT2= 1.25*	**The Lengths notations to determine the normality:** *LTBound1 = 0, LTBound2 = 0* *Core1 = 0, Core2 = 0* *RTBound1 = 0, RTBound2 = 0*



- **Case *CNF.C2*:** When the core and boundary lengths of the fuzzy sets are the same *(KA = KB) if (Ka_i = KA) and (Kb_i = KB)*, the normality of the KH-FRI conclusion fuzzy set is always satisfied. In this case, there is no restriction on the shape and size of the observation A^*. **Table 15** illustrates Example "*CNF.KH_NOR.C2_1*" and Example "*CNF.KH_NOR.C2_2*" to demonstrate ***Case CNF.C2***.

Table 15. *The Normal Conclusion of the KH-FRI with Fuzzy Sets According to Case CNF.C2*

Example ("*CNF.KH_NOR.C2_1*")	
CNF_Notations that prove case *CNF.C2* when (KA = KB)	
The values of the fuzzy sets: A_1= [1 2.5 2.5 4] A_2= [6 7.5 7.5 9] B_1= [1 2.5 2.5 4] B_2= [6 7.5 7.5 9] A^*= [4.5 5 5 5.5] B^*= [4.5 5 5 5.5]	**The case of the Core and (LF and RF) Boundary conclusion:** The length (LFBound) is (*NORMAL*) The length (Core) is (*NORMAL*) The length (RFBound) is (*NORMAL*)
The length ratio between KA, Ka*, and KB: **LTBound**: *RatioLT1= 1, RatioLT2= 1.16* **Core**: *RatioC1 = 1, RatioC2= 1* **RTBound**: *RatioRT1= 1, RatioRT2= 1.16*	**The Lengths notations to determine the normality:** *LTBound1 = 3.5, LTBound2 = 6* *Core1 = 0, Core2 = 0* *RTBound1 = 3.5, RTBound2 = 6*

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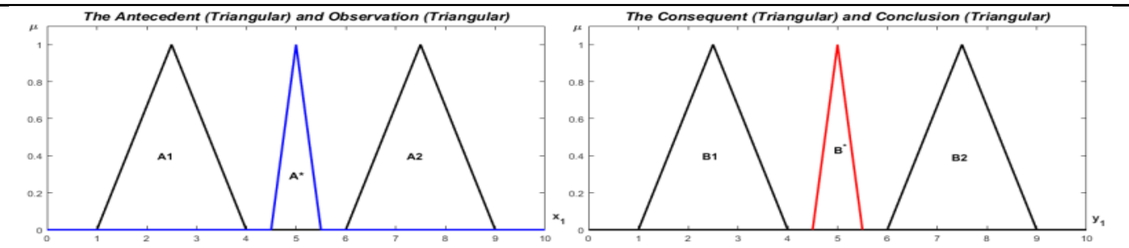
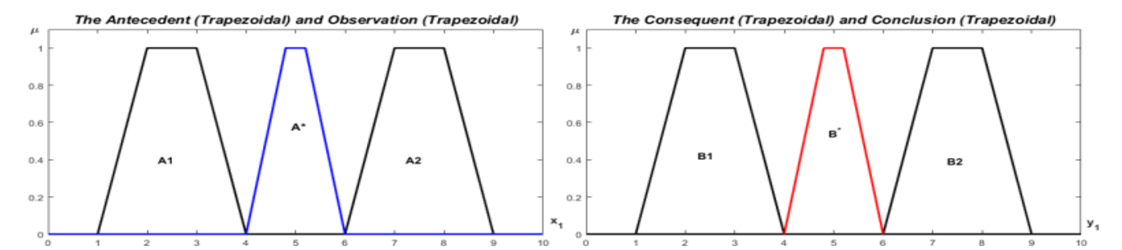


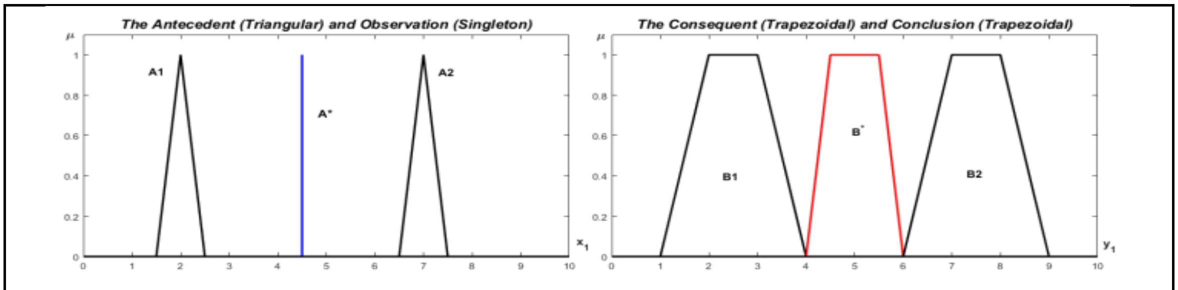
Example ("*CNF.KH_NOR.C2_2*")	
The values of the fuzzy sets: A_1=[1 2 3 4] A_2=[6 7 8 9] B_1=[1 2 3 4] B_2=[6 7 8 9] A^*=[4 4.8 5.2 6] B^*=[4 4.8 5.2 6]	**The case of the Core and (LF and RF) Boundary conclusion:** The length (LFBound) is (*NORMAL*) The length (Core) is (*NORMAL*) The length (RFBound) is (*NORMAL*)
The length ratio between KA, Ka*, and KB: **LTBound**: *RatioLT1= 1, RatioLT2= 1.25* **Core**: *RatioC1 = 1, RatioC2= 1.11* **RTBound**: *RatioRT1= 1, RatioRT2= 1.25*	**The Lengths notations to determine the normality:** *LTBound1 = 0.8, LTBound2 = 4.8* *Core1 = 2.4, Core2 = 4.4* *RTBound1 = 0.8, RTBound2 = 4.8*



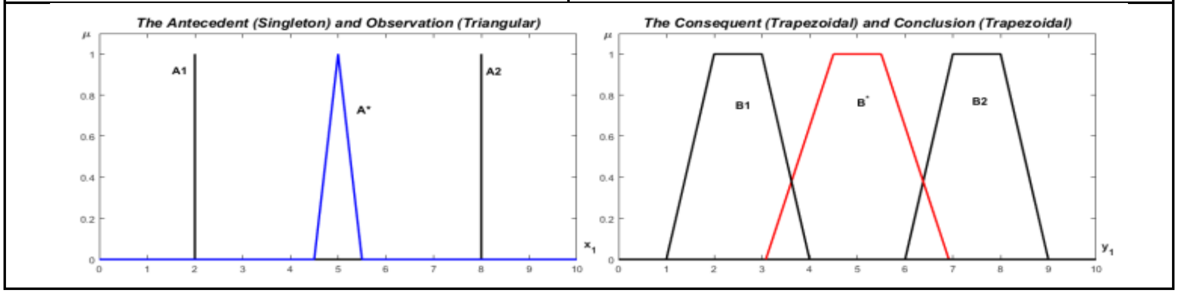
- **Case *CNF.C3*:** If the core and boundary lengths of fuzzy sets *(KB > KA)*, where *(Kai = KA)* and *(Kbi= KB)*, the conclusion of the KH-FRI is always normality. **Table 16** represents Example "*CNF.KH_NOR.C3_1*" and Example "*CNF.KH_NOR.C3_2*" that describes ***Case CNF.C3***.

Table 16. *The Normal Conclusion of the KH-FRI with Fuzzy Sets according to Case CNF.C3*

Example ("*CNF.KH_NOR.C3_1*")	
CNF_Notations that prove case *CNF.C3* when (KB > KA)	
The values of the fuzzy sets: A_1= [1.5 2 2 2.5] A_2= [6.5 7 7 7.5] B_1= [1 2 3 4] B_2= [6 7 8 9] A^*= [4.5 4.5 4.5 4.5] B^*= [4 4.5 5.5 6]	**The case of the Core and (LF and RF) Boundary conclusion:** The length (LFBound) is (*NORMAL*) The length (Core) is (*NORMAL*) The length (RFBound) is (*NORMAL*)
The length ratio between KA, Ka*, and KB: **LTBound**: *RatioLT1= 0.88, RatioLT2= 1* **Core**: *RatioC1 = 0.80, RatioC2= 1* **RTBound**: *RatioRT1= 0.88, RatioRT2= 1*	**The Lengths notations to determine the normality:** *LTBound1 = 2, LTBound2 = 4.5* *Core1 = 0, Core2 = 5* *RTBound1 = 2, RTBound2 = 4.5*



Example ("*CNF.KH_NOR.C3_2*")	
The values of the fuzzy sets: A_1= [2 2 2 2] A_2= [8 8 8 8] B_1= [1 2 3 4] B_2= [6 7 8 9] A^*= [4.5 5 5 5.5] B^*= [3.08 4.5 5.5 6.916]	**The case of the Core and (LF and RF) Boundary conclusion:** The length (LFBound) is (*NORMAL*) The length (Core) is (*NORMAL*) The length (RFBound) is (*NORMAL*)
The length ratio between KA, Ka*, and KB: **LTBound**: *RatioLT1= 0.66, RatioLT2= 1.09* **Core**: *RatioC1= 0.66, RatioC2= 1* **RTBound**: *RatioRT1= 0.66, RatioRT2= 1.09*	**The Lengths notations to determine the normality:** *LTBound1 = -2, LTBound2 = 6.5* *Core1 = 0, Core2 = 6* *RTBound1 = -2, RTBound2 = 6.5*



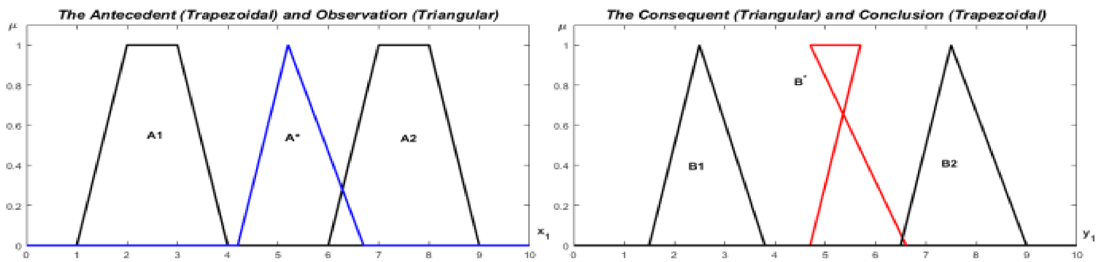
In contrast, the abnormality of the conclusion can appear in **Case *CNF.C4*** *(KB < KA)*. So, to demonstrate the abnormality problem, we will consider the length ratio between Ka^* and *KB* based on **Eq.(6.13)**, **Eq.(6.14)**, and **Eq.(6.15)**. Therefore, we will address the problem with different lengths of core and boundary.

Table 17 - **Table 20** describe the results of **Eq.(6.10)** - **Eq.(6.15)** to prove the normality of the KH-FRI conclusion will not be satisfied. The Example "*CNF.KH_ABNOR.C4_1*" in **Table 17** shows the problem with the core length. Example "*CNF.KH_ABNOR.C4_2*" and Example "*CNF.KH_ABNOR.C4_3*" in **Table 18** and **Table 19** illustrate the problem of the left and right boundary. The Example "*CNF.KH_ABNOR.C4_4*" in **Table 20** shows the problem in both the core and boundary lengths.

Table 17 describes the abnormality in the core length of the KH-FRI conclusion.

Table 17. *The Problem with Core Length, Abnormal Conclusion*

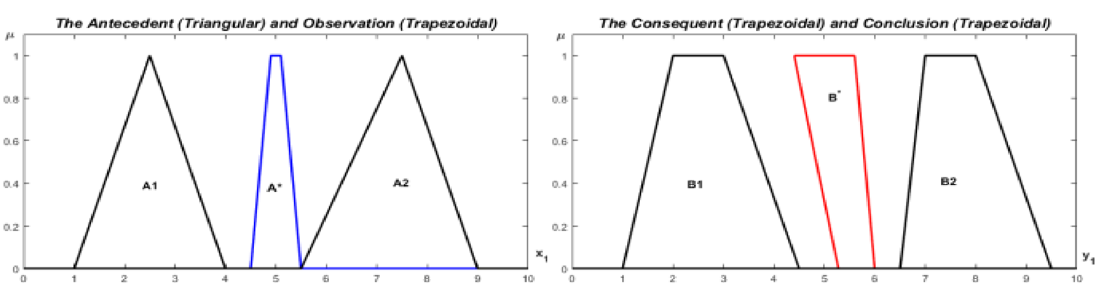
Example ("CNF.KH_ABNOR.C4_1")	
The values of the fuzzy sets: A_1=[1 2 3 4] A_2=[6 7 8 9] B_1= [1.5 2.5 2.5 3.8] B_2= [6.5 7.5 7.5 9] A^*=[4.2 5.2 5.2 6.7] B^*= [4.7 5.7 4.7 6.6]	**The case of the Core and (LF and RF) Boundary conclusion:** The length (LFBound) is (<u>*NORMAL*</u>) The length (Core) is (<u>***PROBLEM***</u>) The length (RFBound) is (<u>*NORMAL*</u>)
The length ratio between KA, Ka^*, and KB: **LTBound**: *RatioLT1= 1, RatioLT2= 1.33* **Core**: *RatioC1 = 1.25, RatioC2= 1* **RTBound**: *RatioRT1= 0.92, RatioRT2= 1.6*	**The Lengths notations to determine the normality:** *LTBound1 = 0, LTBound2 = 5* *Core1 = 5, Core2 = 0* *RTBound1 = -9.25, RTBound2 = 17.28*



An explanation of the abnormality of the left boundary length in the KH-FRI conclusion is shown in **Table 18**.

Table 18. *The Problem with Left Length, Abnormal Conclusion*

Example ("*CNF.KH_ABNOR.C4_2*")	
The values of the fuzzy sets: A_1=[1 2.5 2.5 4] A_2=[5.5 7.5 7.5 9] B_1= [1 2 3 4.5] B_2= [6.5 7 8 9.5] A^*=[4.5 4.9 5.1 5.5] B^*= [5.27 4.4 5.6 6.0]	**The case of the Core and (LF and RF) Boundary conclusion:** The length (LFBound) is (<u>***PROBLEM***</u>) The length (Core) is (<u>*NORMAL*</u>) The length (RFBound) is (<u>*NORMAL*</u>)
The length ratio between KA, Ka^*, and KB: ***LTBound***: *RatioLT1 = 1.5, RatioLT2= 1.15* ***Core***: *RatioC1= 0.8, RatioC2= 1.04* ***RTBound***: *RatioRT1= 1, RatioRT2= 1.12*	**The Lengths notations to determine the normality:** *LTBound1 = 30.15, LTBound2 = 6.80* *Core1 = -0.8, Core2 = 5.2* *RTBound1 = 3.85, RTBound2 = 5.85*



An illustration of the abnormality in the right boundary length in the KH-FRI conclusion is displayed in **Table 19**.

Table 19. *The Problem with Right Length, Abnormal Conclusion*

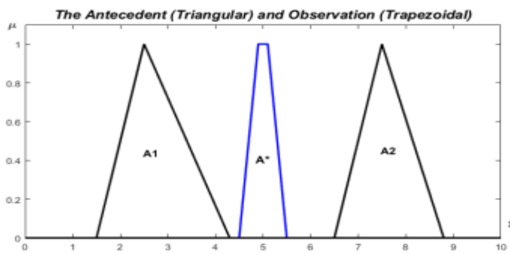
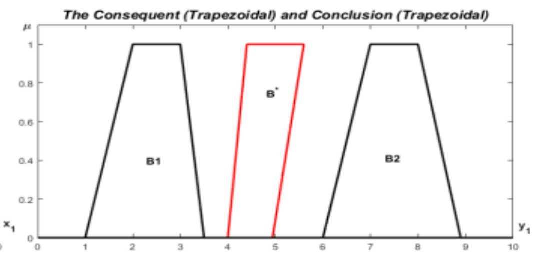
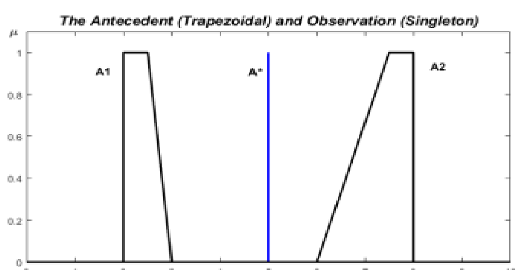
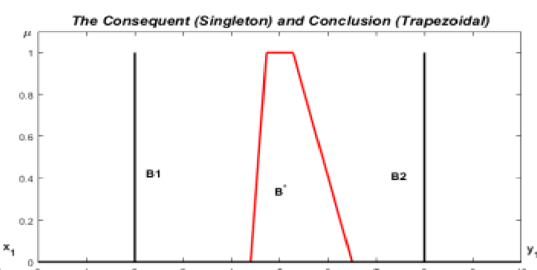
Example ("*CNF.KH_ABNOR.C4_3*")	
The values of the fuzzy sets: A_1=[1.5 2.5 2.5 4.3] A_2=[6.5 7.5 7.5 8.8] B_1= [1 2 3 3.5] B_2= [6 7 8 8.9] A^*=[4.5 4.9 5.1 5.5] B^*= [4 4.4 5.6 4.94]	**The case of the Core and (LF and RF) Boundary conclusion:** The length (LFBound) is (*NORMAL*) The length (Core) is (*NORMAL*) The length (RFBound) is (***PROBLEM***)
The length ratio between KA, Ka*, and KB: ***LTBound***: *RatioLT1 = 1, RatioLT2= 1.11* ***Core***: *RatioC1= 0.80, RatioC2= 1.04* ***RTBound***: *RatioRT1= 1.40, RatioRT2= 1.14*	**The Lengths notations to determine the normality:** *LTBound1 = 2.4, LTBound2 = 4.4* *Core1 = -0.8, Core2 = 5.2* *RTBound1 = 25.65, RTBound2 = 6.76*
	

Table 20 describes the abnormality in both core and boundary lengths in the KH-FRI conclusion.

Table 20. *The Problem with Core and Boundary Lengths, Abnormal Conclusion*

Example ("*CNF.KH_ABNOR.C4_4*")	
The values of the fuzzy sets: A_1=[2 2 2.5 3] A_2=[6 7.5 8 8] B_1=[2 2 2 2] B_2=[8 8 8 8] A^*=[5 5 5 5] B^*=[6.5 5.27 4.72 4.4]	**The case of the Core and (LF and RF) Boundary conclusion:** The length (LFBound) is (***PROBLEM***) The length (Core) is (***PROBLEM***) The length (RFBound) is (***PROBLEM***)
The length ratio between KA, Ka*, and KB: ***LTBound***: *RatioLT1 = 1.5, RatioLT2= 1* ***Core***: *RatioC1= 1.2, RatioC2= 1* ***RTBound***: *RatioRT1= 1.2, RatioRT2= 1*	**The Lengths notations to determine the normality:** *LTBound1 = 27, LTBound2 = 0* *Core1 = 3, Core2 = 0* *RTBound1 = 9, RTBound2 = 0*
	

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6.2.6. Discussion of the CNF benchmark examples

The cases and equations were discussed earlier have been used to construct *CNF_Benchmark Examples*. The *CNF_Benchmark Examples* are classified into two groups, as shown in **Table 14 - Table 20.** The first group contains Examples "*CNF.KH_NOR.C1*" – "*CNF.KH_NOR.C3_2*" described the normality conclusion of the KH-FRI, where Corollaries of the normality condition (Corollaries *CNFCR1*– *CNFCR4*) are met. The second group includes Examples "*CNF.KH_ABNOR.C4_1*" – "*CNF.KH_ABNOR.C4_4*" described the abnormality conclusion of the KH-FRI. The abnormality could appear in case the corollary $(KB < KA)$, *if* $(KA^* < KB)$.

Referring to Example "*CNF.KH_NOR.C1*" as shown in Table 14, we conclude the following:

- The core and boundary lengths of fuzzy sets of the observation (KA^*) is large or equal than the antecedents (KA) (see case *CNF.C1*) and if $(Ka_i = KA)$, then the conclusion is always normal. In this case, there is no restriction on the shape and size of the consequent (KB).
  - E.g., the core lengths of the antecedent fuzzy sets are ($KA_1= 0$ *(2-2)*, $KA_2=0$ *(8-8)*, and the average length of the antecedent fuzzy sets $KA=0$ *(5-5)*), and for the observation fuzzy set ($KA^*=0$).
- Regarding **Eq.(6.16)** and **Eq.(6.17)**, all CNF_Notations satisfy the normality conclusion, i.e.:
  - The core length (Core1=0) is less than or equal to (*Core2=0*),
  - The left length (*LTBound1=0*) is also less than (*LTBound2=0*).
  - The right length (*RTBound1=0*) is also less than (*RTBound2=0*).
- From another side, the length ratio using **Eq.(6.13)**, **Eq.(6.14)**, and **Eq.(6.15)** are also satisfied with the normality, i.e.:
  - The length ratio of the core (*RatioC1= 1*) is less than or equal (*RatioC2= 1*).
  - The length ratio of the left boundary (*RatioLT1= 1.20*) is less than (*RatioLT2= 1.25*).
  - The length ratio of the right boundary (*RatioRT1= 1.20*) is less than (*RatioRT2= 1.25*).
- In this case $(KA>=KA^*)$ and if ($Ka_i = KA$), the conclusion $B^*= [5\ 5\ 5\ 5]$ is always normal.

In the Example "*CNF.KH_NOR.C2_2*" as shown in Table 15, we conclude the following:

- The core and boundary lengths of fuzzy sets antecedents (KA) are equal consequents (KB) (see case *CNF.C2*), and if $(Ka_i= KA)$, $(Kb_i= KB)$, then the conclusion is always normal. In this case, there is no restriction on the length of the observation (KA^*) fuzzy set.
  - E.g., the right lengths of the antecedents fuzzy sets are ($KA_1=1$ *(4-3)*, $KA_2=1$ *(9-8)*, and the average length of the antecedent fuzzy sets $KA=1$), and the consequents fuzzy

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sets are (KB_1=1 (4-3), KB_2=1 (9-8), and the average length of the consequents fuzzy sets *KB=1*).

- The normality conclusion is satisfying, according to **Eq.(6.16)**. I.e.:
  - The core length (*Core1=4.2*) is less than (*Core2=4.4*).
  - The left length (*LTBound1=0.8*) is less than (*LTBound2=4.8*).
  - The right length (*RTBound1=0.8*) is less than (*RTBound2=4.8*).
- Also, according to the length ratio using **Eq.(6.13)**, **Eq.(6.14)**, and **Eq.(6.15)** are also satisfied with core and boundary, i.e.:
  - The length ratio of the core (*RatioC1= 1*) is less than (*RatioC2= 1.11*).
  - The length ratio of the left boundary (*RatioLT1= 1*) is less than (*RatioLT2= 1.25*).
  - The length ratio of the right boundary (*RatioRT1= 1*) is less than (*RatioRT2= 1.25*).
- In this case *(KA=KB)*, and if (Ka_i= KA), (Kb_i= KB), the conclusion B^*= *[4 4.8 5.2 6]* is always normal.

For Example "*CNF.KH_NOR.C3_1*" as shown in Table 16, we conclude the following:

- The core and boundary lengths of fuzzy sets consequents (*KB*) are large than antecedents (*KA*) (see case *CNF.C3*), where *(Kai = KA)* and *(Kbi= KB)*, then the conclusion is always normal.
  - E.g., the left lengths of the consequents fuzzy sets are (KB_1=1 (2-1), KB_2=1 (7-6), and the average length of the consequents fuzzy sets *KB=1*). The antecedents fuzzy sets are (KA_1=0.5 (2-1.5), KA_2=0.5 (7-6.5). The average length of the antecedent fuzzy sets *KA=0.5*).
- The normality conclusion is satisfying according to **Eq.(6.17)**, i.e.:
  - The core length (*Core1=0*) is less than (*Core2=5*).
  - The left boundary length, (*LTBound1 = 2*) is less than (*LTBound2 = 4.5*), the conclusion is normal.
  - The right boundary length, (*RTBound1 = 2*) is less than (*RTBound2 = 4.5*), the conclusion is normal.
- Also, according to the length ratio using **Eq.(6.13)**, **Eq.(6.14)**, and **Eq.(6.15)**, are also satisfied for core and boundary, i.e.:
  - The length ratio of the left boundary LTBound: (*RatioLT1= 0.88*) is less than (*RatioLT2= 1*),
  - The length ratio of the core: (*RatioC1= 0.80*) is less than (*RatioC2= 1*),
  - The length ratio of the right boundary RTBound: (*RatioRT1= 0.88*) is less than (*RatioRT2= 1*).
- In this case *(KB>KA)*, where *(Kai = KA)* and *(Kbi= KB)*, the conclusion B^*= *[4 4.5 5.5 6]* is always normal,

Nevertheless, the second group includes Examples "*CNF.KH_ABNOR.C4_1*" – "*CNF.KH_ABNOR.C4_4*", the abnormality conclusion could appear in case the length of the consequence (KB) is less than the length of the antecedents (KA) with taking into consideration the length ratio $(Ka^* < KA)$. Therefore, all CNF_Notations used to check the normality conclusion are proved these examples not satisfied with CNF property. **Eq.(6.13) - Eq.(6.15)** are important to prove abnormality in the core and boundary conditions (e.g., in case the core length, when ratio $[Core(KA)/Core(KB))]$ does not exceed ratio $[Core(KA)/(Core(Ka^*.KA_1) + Core(KA_2.Ka^*))])$.

In Example "*CNF.KH_ABNOR.C4_1*" as shown in Table 17, we conclude the following:

- The core and boundary lengths of the fuzzy sets consequents (KB) are less than antecedents (KA). The observation fuzzy set is less than consequent fuzzy sets $(KA^* < KB)$ (see case *CNF.C4*), then the conclusion is always abnormal.
  - I.e., the core lengths of the consequents fuzzy sets are ($KB_1=0$ $(2.5\text{-}2.5)$, $KB_2=0$ $(7.5\text{-}7.5)$, and the average length of the consequents fuzzy sets $KB=0$), and the antecedents' fuzzy sets are ($KA_1=1$ $(3\text{-}2)$, $KA_2=1$ $(8\text{-}7)$. The average length of the antecedent fuzzy sets $KA=1$), and the core length of the observation fuzzy set $KA^*=0$ $(5.2\text{-}5.2)$.
- For the length ratio CNF_Notations used to prove the problem (abnormality in the core), using (**Eq.(6.14)**) as follows:
  - The length ratio of core is not satisfied, as ($RatioC1 = 1.25$) exceeds the ($RatioC2 = 1$).
- From another side, **Eq.(6.17)** also demonstrates an issue for the core length.
  - The core length values, ($Core1 = 5$) is greater than ($Core2 = 0$).
- In this case $(KB<KA)$ *and* $(KA^* < KB)$, the conclusion $B^* = [4.7\ 5.7\ 4.7\ 6.6]$ is always abnormal, as shown by values of the core (5.7 is greater than 4.7).

In Example "*CNF.KH_ABNOR.C4_2*" as shown in Table 18, we conclude the following:

- The core and boundary lengths of the fuzzy sets $KB < KA$ and $KA^* < KB$. In this case, the conclusion $B^* = [5.27\ 4.4\ 5.6\ 6.0]$ is always abnormal, where the problem could appear in the left values as (5.27 is greater than 4.4).
  - I.e., the left lengths of the consequents fuzzy sets are ($KB_1=1$ $(2\text{-}1)$, $KB_2=0.5$ $(7\text{-}6.5)$, and the average length of the consequents fuzzy sets $KB=0.75$). The antecedents fuzzy sets are ($KA_1=1.5$ $(2.5\text{-}1)$, $KA_2=2$ $(7.5\text{-}5.5)$, and the average length of the antecedent fuzzy sets $KA=1.75$), and the left length of the observation fuzzy set $KA^*=0.4$ $(4.9\text{-}4.5)$.
- For the length ratio CNF_Notations used to prove the problem (abnormality in the core), using (**Eq.(6.13)**) as follows:
  - The length ratio of the left boundary is not satisfied, where ($RatioLT1 = 1.5$) exceeds the ($RatioLT2 = 1.15$).
- Also, using **Eq.(6.10)** demonstrates a problem as follows:
  - The left length values, where ($LTBound1 = 30.15$) is greater than ($LTBound2 = 6.80$), the conclusion is suffering from the abnormality.

In Example "*CNF.KH_ABNOR.C4_3*" as shown in Table 19, we conclude the following:

- The core and boundary lengths of $KB < KA$ and $KA^* < KA$. In this case, the conclusion $B^*= [4\ 4.4\ 5.6\ 4.94]$ is always abnormal, where the problem could appear in the right values (5.6 is greater than 4.94).
- I.e., the right lengths of the consequents fuzzy sets are ($KB_1=0.5\ (3.5\text{-}3)$, $KB_2=0.9\ (8.9\text{-}8)$, and the average length of the consequents fuzzy sets $KB=0.7$). The antecedents fuzzy sets are ($KA_1=1.8\ (4.3\text{-}2.5)$, $KA_2=1.3\ (8.8\text{-}7.5)$. The average length of the antecedent fuzzy sets $KA=1.55$), and the right length of the observation fuzzy set $KA^*=0.4\ (5.5\text{-}5.1)$.
- For the length ratio CNF_Notations used to prove the problem (abnormality in the core), using **Eq.(6.15)** as follows:
  - The length ratio of the right boundary is not satisfied, where (*RatioRT1= 1.40*) exceeds (*RatioRT2= 1.14*).
- Also, using **Eq.(6.12)** demonstrates a problem as follows:
  - The left length values, where (*RTBound1= 25.65*) is greater than (*RTBound1= 6.76*), and therefore, the conclusion is suffering from the abnormality.

In Example "*CNF.KH_ABNOR.C4_4*" as shown in Table 20, we conclude the following:

- We can see the problems in both of the core and boundary lengths, where the conclusion $B^*= [6.5\ 5.27\ 4.72\ 4.4]$ is always abnormal.
- Regarding the CNF_Notations by **Eq.(6.10)**, **Eq.(6.11)**, **Eq.(6.12)**, and **Eq.(6.17)** of the left and right boundary are not satisfied with the normality:
  - Left Boundary length**:** *LTBound1=27 > LTBound2=0*,
  - Core length**:** *Core1=3 > Core2=0*,
  - Right Boundary length**:** *RTBound1=9 > RTBound2=0*.
- Also, **Eq.(6.13) - Eq.(6.15)** are not satisfied because (Ratio1) is greater than (Ratio2) for the core and boundary of the conclusion:
  - Left ratio: *RatioLT1=1.5 > RatioLT2=1*
  - Core: *RatioC1=1.2 > RatioC2=1*
  - Right ratio: *RatioRT1=1.2 > RatioRT2=1*

Based on all examples and cases of the CNF property benchmark, we conclude the following:

1. Regarding the examples of the first group. We conclude that all examples are satisfied with CNF property according to the difference between the core and boundary conditions of the fuzzy rules and observation fuzzy sets. Where the conclusion is always "normal" in case:
   - $(KA^* >= KA)$, *if* $(Ka_i = KA)$.
   - $(KA = KB)$, *if* $(Ka_i = KA)$ *and* $(Kb_i = KB)$.
   - $(KB > KA)$, *if* $(Ka_i = KA)$ *and* $(Kb_i = KB)$

Moreover, all CNF_Notations in **Eq.(6.10) - Eq.(6.15)** of the core and Left-Right boundary (lengths and ratios) proved that the *normality* conclusion, which is always *satisfied* with the CNF condition.

2. On the other hand, regarding the examples of the second group. We conclude that all examples are not satisfied with CNF property according to the difference between the core and boundary conditions of the fuzzy rules and observation, where the conclusion is always abnormal in case:
   - $(KB < KA)$, *if* $(KA^{*} < KB)$.

Furthermore, all CNF_Notations in **Eq.(6.10) - Eq.(6.15)** of the core and Left-Right boundary (lengths and ratios) proved that the *abnormality* conclusion, which is always *not satisfied* with the CNF condition.

6.3. <u>FRI Benchmark Example of the PWL Property for the Koczy-Hirota Interpolation Method</u>

Among several conditions for fuzzy interpolation techniques which were recommended in [15], [17] and [43], the conservation of the Piecewise Linearity (PWL) is an important property for reducing the computational complexity, in case of α-cut based FRI methods (like the studied KH-FRI). If the FRI method preserves the Piecewise Linearity of the fuzzy sets, and all the fuzzy values, and the observation are PWL fuzzy sets, the FRI calculations can be reduced some related α-cut levels.

The most significant benefit of the KH-FRI is its low computational complexity. Notwithstanding the advantages, in some antecedent fuzzy set configuration, the KH-FRI conclusion suffers from preserving a PWL (for more details see [4], [5]). The preservation of PWL is a significant property to be able to eliminate the FRI calculations between two consecutive α-cuts. The studies in [41], [44], and [45] discuss the PWL property and gives some conditions for the fuzzy rules and observation fuzzy sets, where the PWL of the conclusion necessarily holds.

This subsubsection aims to highlight the problematic properties of the KH-FRI method to prove its efficiency with PWL condition to construct *PWL_Benchmark Examples*. This benchmark can serve as a baseline for testing other FRI methods against cases that the KH-FRI is not satisfied with the linearity condition. All benchmark examples in this subsubsection are constructed using notations and equations detailed in [41], [44], [45], and implemented in the MATLAB FRI Toolbox [24], [26], which provides an easy-to-use framework to represent the conclusions of the FRI methods.

6.3.1. The shape of the KH-FRI conclusion in case of PWL fuzzy sets

The main concept of the KH-FRI is based on the resolution and extension principles [90], in which the FRI can be decomposed to the problem into an infinite family of crisp issues corresponding to α-cuts of fuzzy rule bases and observation. The interpolation conclusion can be solved for every α-cuts independently. And it can deduce the fuzzy solution by combining these results into a fuzzy approximation (see **Eq.(6.8)**).

Most FRI methods require some constraints to be satisfied: all the fuzzy sets of fuzzy rules and observation must be convex and normal, or briefly a CNF set. Let us assume (A) is a fuzzy set; thus, (A) is called normal when ($Height(A) = max(x) \in U(\mu_A(x))$), and is convex if each of its α-cuts is connected. Thus, the Membership Functions (*MF*) of fuzzy rules and observation (e.g., trapezoidal and triangular) are also restricted to be *PWL* because it will be much easier for calculation with such functions because it depends on α-cuts. Definitions (*Definition of the Preceding CNF Sets - Definition of the Lower and Upper Distances Between α-cuts in previous subsection 6.2*) could be introduced to realize the interpolation concept.

Fig. 78 represents the linear interpolation method between two fuzzy rule bases and observation described by trapezoidal Membership Function (*MF*) for $\alpha \in [0, 1]$. The characteristic points of the trapezoidal MF denoted by vector $a= [a_1, a_2, a_3, a_4]$, where the support (a_1 and a_4) represents by $P\ (0, L)$ and $P\ (0, U)$, the core (a_2 and a_3) describes by $P\ (1, L)$ and $P\ (1, U)$, in which L denotes to lower, and U denotes to upper. In the case of triangular *MF*, it can be represented by $P\ (0, L)$, $P\ (0, U)$, and $P\ (1, (L \text{ and } U))$, in which $a_2=a_3$ for the core fuzzy set A.

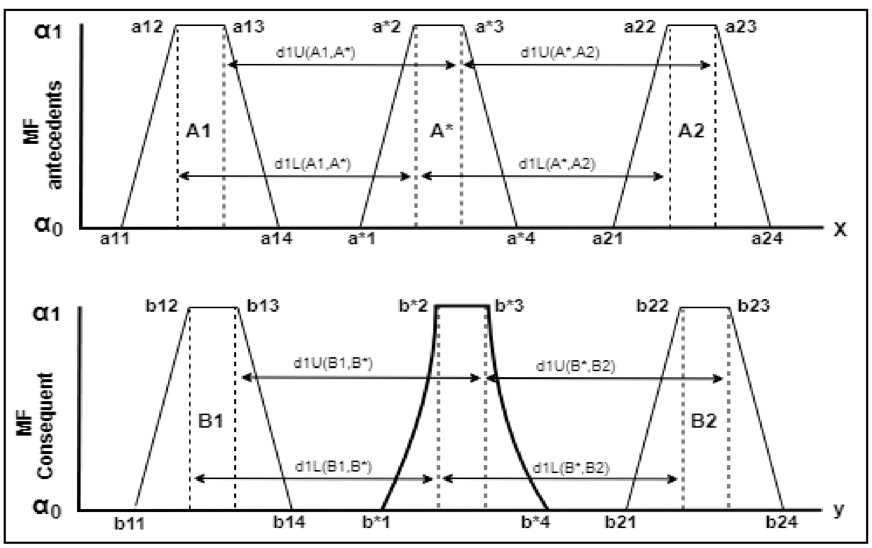

Fig. 78. *The Ratio of the Lower and Upper Distances Calculated Between the Interpolation of Two Piecewise Linear Rules. The Shape of the Conclusion (B^*) Shows for the α-Cuts Level Between $\alpha \in (0, 1)$ [41]*

6.3.2. The piecewise linearity of the KH-FRI conclusion

Many FRI methods are not preserving the PWL in conclusion (see cases in [32]). The KH-FRI is one of them, which also cannot fulfill this condition. Most of the FRI methods, which hold the PWL condition, the FRI calculations are restricted to a small finite set of α-cut levels, which will be called the necessary cuts. For PWL membership functions (e.g., trapezoidal and triangular), an obvious assumption is to define the set of significant cuts by the united breakpoint set α. However, this is not true in general because in many FRI methods, the conclusion B^* is severely distorted and non-linear (i.e., the PWL condition not holds).

Theoretically, the conclusion of the KH-FRI can be calculated by its α-cuts. All α-cuts should be considered, but for practical reasons, only a finite set is taken into consideration during the computation. Now, let us determine which PWL_Notations will be used to calculate the characteristic points of the lower and upper of fuzzy rule bases and observation fuzzy sets that can be defined as follows:

For antecedent fuzzy set:

$$A_{i\alpha L} = \alpha \,.\, (a_{i2} - a_{i1}) + a_{i1}$$
$$A_{i\alpha U} = \alpha \,.\, (a_{i3} - a_{i4}) + a_{i4} \tag{6.18}$$

For consequent fuzzy set:

$$B_{i\alpha L} = \alpha \,.\, (b_{i2} - b_{i1}) + b_{i1}$$
$$B_{i\alpha U} = \alpha \,.\, (b_{i3} - b_{i4}) + b_{i4} \tag{6.19}$$

For observation fuzzy set:

$$A^*_{\alpha L} = \alpha \,.\, (a^*_2 - a^*_1) + a^*_1$$
$$A^*_{\alpha U} = \alpha \,.\, (a^*_3 - a^*_4) + a^*_4 \tag{6.20}$$

The conclusion of the linear interpolation (left slope) could be calculated for α-cut levels by the statement as follows:

Statement *PWLST1*: The equations of the left and right slopes to breakpoint levels 0 and α can be calculated for the two fuzzy rule bases $A_1 \rightarrow B_1$, $A_2 \rightarrow B_2$, and the observation A^* as follows:

$$B^*_{\alpha L} = \frac{DL_1 \times \alpha^2 + DL_2 \times \alpha + DL_3}{cL_9 \times \alpha + cL_{10}} \tag{6.21}$$

$$B^*_{\alpha R} = \frac{DR_1 \times \alpha^2 + DR_2 \times \alpha + DR_3}{cR_9 \times \alpha + cR_{10}} \tag{6.22}$$

where

$$DL_1 = (cL_3.cL_5) + (cL_1.cL_7)$$
$$DL_2 = (cL_3.cL_6) + (cL_4.cL_5) + (cL_1.cL_8) + (cL_2.cL_7)$$
$$DL_3 = (cL_4.cL_6) + (cL_2.cL_8)$$

And

$$cL_1 = a^*_2 - a^*_1 - a_{12} + a_{11};\ cL_2 = a^*_1 - a_{11};\ cL_3 = a_{22} - a_{21} - a^*_2 + a^*_1$$
$$cL_4 = a_{21} - a^*_1;\ cL_5 = b_{12} - b_{11}; cL_6 = b_{11};\ cL_7 = b_{22} - b_{21}$$
$$cL_8 = b_{21}; cL_9 = a_{11} - a_{12} + a_{22} - a_{21};\ \ cL_{10} = a_{21} - a_{11}$$

Similar to the left slope equation, the right slope can be constructed, it can replace the index *1* of the characteristic points fuzzy set $a_{1(1)}$, $a_{2(1)}$, $a^*_{(1)}$, $b_{1(1)}$ and $b_{2(1)}$ by index *4*, and index *2* of $a_{1(2)}$, $a_{2(2)}$, $a^*_{(2)}$, $b_{1(2)}$ and $b_{2(2)}$ are replaced by *3*, and the sign in *X* replaced by its opposite (negative direction tangents).

On the other hand, authors in [41], [44], [45] also introduced other equations to calculate the left and right slopes of the conclusion as follows:

The left slope of the conclusion:

$$B_L^* = \frac{(DL_3.cL_9^2) - (DL_2.cL_9.cL_{10}^2) + (DL_1.cL_{10}^2)}{cL_9^2.(cL_9.\alpha + cL_{10})} \times \frac{DL_1}{cL_9}.\alpha + \frac{(DL_2.cL_9) - (DL_1.cL_{10})}{cL_9^2} \tag{6.23}$$

it can be written:

$$\frac{A}{\alpha + B} + (C.\alpha + D) = yH + yL \tag{6.24}$$

where

$$A = \frac{(DL_3.cL_9^2) - (DL_2.cL_9.cL_{10}^2) + (DL_1.cL_{10}^2)}{cL_9^3}$$

$$B = \frac{cL_{10}}{cL_9},\ C = \frac{DL_1}{cL_9},\ D = \frac{(DL_2.cL_9) - (DL_1.cL_{10})}{cL_9^2}$$

Where y_L refers to a straight line and y_H denotes the hyperbola, $B_{\alpha L}$ is the curve that represents the superposition of y_L and y_H (for more details see Figure (11) in [41], [45]). Similar equations of the left slope can calculate the right slope.

6.3.3. Reference values for the PWL property

The KH-FRI conclusion is not preserving the PWL property, becuase the rate calculated by the KH fundamental equation between two adjacent fuzzy rules and the observation, which are changing for all the α levels. According to the main corollaries in [41], [44], [45], the linearity of the left and right slopes of the KH-FRI conclusion could be determined as follows:

The condition of polynomiality when ($cL_9 = 0$). Then, we get:

1) Corollary *PWLCR1*:

The flanks of B^* are piecewise polynomial if and only if the two antecedents A_1 and A_2 have equivalent PWL slopes, obtainable from each other by geometric translations:

$$a_{12} - a_{11} = a_{22} - a_{21} \tag{6.25}$$

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If we require linearity of the pieces, the condition must be met, when ($DL_1 = 0$). Consequently, the linearity conclusion can be demonstrated:

2) Corollary *PWLCR2*:

This corollary will be satisfied in three different cases that slopes of the conclusion B^* are preserving PWL. Hence, if this corollary is done suitably, the KH-FRI conclusion will always be satisfied if the following cases are held:

2.1) Case *PWLC1*:

If the left and right slopes of the antecedents A_i and the consequents B_i are equivalent to PWL on the universe of discourse, then the left slope PWL_Notations can be defined as:

$$\begin{aligned} A_i &= a_{12} - a_{11} = a_{22} - a_{21} \\ B_i &= b_{12} - b_{11} = b_{22} - b_{21} \end{aligned} \tag{6.26}$$

2.2) Case *PWLC2*:

If the left and right slopes and characteristic points of the two adjacent fuzzy rule bases $A_1 \Rightarrow B_1$ and $A_2 \Rightarrow B_2$ are equivalent in the universe of discourse, then the left slope PWL_Notations could be determined as follows:

$$\begin{aligned} A_1 \Rightarrow B_1&: a_{12} - a_{11} = b_{12} - b_{11} \\ A_2 \Rightarrow B_2&: a_{22} - a_{21} = b_{22} - b_{21} \end{aligned} \tag{6.27}$$

In this case, there is no restriction on the shape of observation A^*.

2.3) Case *PWLC3*:

If the antecedents A_i and the observation A^* are satisfied with PWL. The B^* slopes are linear only if Corollary *PWLCR1* is applied.

The left slope PWL_Notations can be determined as follows:

$$d = d^* \tag{6.28}$$

where

$$a_{22} - a_{21} = a_{21} - a_{11} = d$$
$$a^*_2 - a^*_1 = d^*$$

For this case, there is no restriction on the consequents B_i.

2.4) Case *PWLC4*:

If all the variables on the universe of discourse are covered by equidistant fuzzy sets A_i, B_i, and A^*, then PWL_Notations of the left slope can be described as follows:

$$\begin{aligned} A_i &= a_{12} - a_{11} = a_{22} - a_{21} \\ B_i &= b_{12} - b_{11} = b_{22} - b_{21} \end{aligned} \tag{6.29}$$

$$A^* = a^*_2 - a^*_1$$

In [41], [44], [45], the upper bound is presented the possible highest deviation between the real and approximated linear functions, hence, if there is a vast difference between them, the validity of the method is violated between characteristic points of the fuzzy sets in the interval [0, 1], and at the same time could question the applicability of any new method. Regarding the beneficial computational properties of the KH-FRI would not hold anymore. Consequently, different views were introduced to determine the deviation from the calculated linear interpolation. Therefore, the approximating linear equation of the conclusion defined to give a straight line that will be used to compare with real function. It can be determined as follows:

For the left slope of conclusion B^* for two endpoints [0, 1] are:

$$B^*_{0\mathrm{L}} = \frac{DL_3}{DL_{10}}, \; B^*_{1\mathrm{L}} = \frac{DL_1 + DL_2 + DL_3}{cL_9 + cL_{10}} \tag{6.30}$$

Then, the equation of the left slope of the linear approximation is determined as:

$$B^*_{\alpha L(approx)} = \alpha \times (B^*_{1L} - B^*_{0L}) + B^*_{0L} \tag{6.31}$$

Fig. 79 describes the maximum difference between the real function and its PWL approximation, which can be determined by statement *PWLST2*:

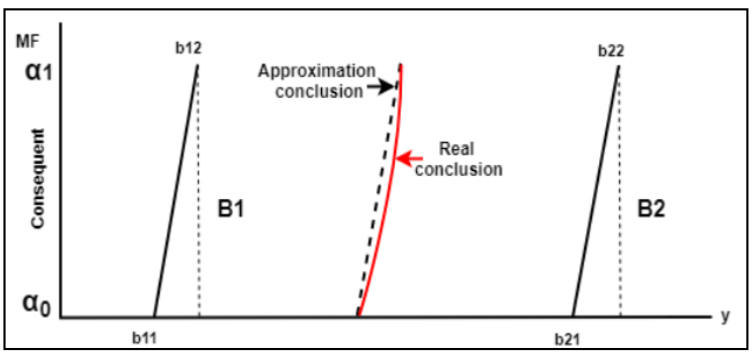

Fig. 79. *The Difference Between the Linear Approximation and Real Function of the Left Slope for the α-Cuts Level Between α∈(0,1) [41], [45]*

Statement *PWLST2*: The error of the approximating nonlinear slope of the determined conclusion could be defined by a linear slope between (*0* and *1*), which expressed in terms of the membership degree running through [0, 1] as follows:

$$\Delta B^*_{\alpha L} = \frac{DL_1}{cL_9 \times \alpha + cL_{10}} \times \alpha^2 + \tag{6.32}$$

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$$\left(\frac{DL_2}{cL_9 \times \alpha + cL_{10}} + \frac{DL_3}{cL_{10}} - \frac{DL_1 + DL_2 + DL_3}{cL_9 + cL_{10}}\right) \times \alpha + \left(\frac{DL_3}{(cL_9 \times \alpha) + cL_{10}} - \frac{DL_3}{cL_{10}}\right)$$

In **Eq.(6.23)** could be used to verify the PWL condition, and for further PWL_Notations presented to check the upper limit of the error can be given by calculating the difference $y_H(0) - y_H(1)$ (for more details see [41], [45]), which can be determined as follows:

$$E = yH(0) - yH(1) = \frac{A}{B(1+B)} \tag{6.33}$$
$$\frac{(DL_3.cL_9^2) - (DL_2.cL_9.cL_{10}) + (DL_1.cL_{10}^2)}{cL_9.cL_{10}.(cL_9 + cL_{10})}$$

Where, the linearity error can be determined by Statement *PWLST3*:

Statement *PWLST3*: The linearity error of B^*_L (for the left slope) does not exceed $\varepsilon > 0$ if:

$$\frac{(DL_2 + cL_{10}.\varepsilon) + \sqrt{(DL_2 + cL_{10}.\varepsilon)^2 - 4.DL_1.(DL_3 - cL_{10}.\varepsilon)}}{2.(DL_3 - cL_{10}.\varepsilon)}$$
$$\leq \frac{cL_9}{cL_{10}} = \frac{1}{B} \tag{6.34}$$
$$\frac{(DL_2 + cL_{10}.\varepsilon) - \sqrt{(DL_2 + cL_{10}.\varepsilon)^2 - 4.DL_1.(DL_3 - cL_{10}.\varepsilon)}}{2.(DL_3 - cL_{10}.\varepsilon)}$$

Which are proved by:
The left slope:

$$Left.slope = ((DL_3 - (cL_{10} \times \varepsilon)) \times cL_9^2) - ((DL_2 \times cL_{10} + cL_{10}^2 \times \varepsilon) \times cL_9 + (DL_1 \times cL_{10}^2)) <= 0 \tag{6.35}$$

The right slope:

$$Right.slope = ((DR_3 - (cR_{10} \times \varepsilon)) \times cR_9^2) - ((DR_2 \times cR_{10} + cR_{10}^2 \times \varepsilon) \times cR_9 + (DR_1 \times cR_{10}^2)) <= 0 \tag{6.36}$$

The value ε is assumed 0 to verify PWL_Notations of the statement *PWLST3*.

The general case of the linear interpolation can only use two breakpoint values ($\alpha = 0$ and $\alpha = 1$) for computing the support and the core conclusion, which may not be satisfactory because, in most cases, the results obtained are somewhat disappointing. For this reason, it will be needed to calculate for a much larger number of α-cuts levels. In the next subsubsection, we will discuss

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all cases used in constructing the *PWL_Benchmark Examples*. These cases will be analyzed according to the PWL condition, which values of α-cut levels to every step of 0.1, $\alpha \in [0, 1]$ will be considered.

6.3.4. The KH-FRI PWL benchmark

In this subsubsection, the validity of the PWL condition of the KH-FRI method will be investigated. The statements and equations in the previous sub subsections could be used to check the linearity of the KH-FRI conclusion, and also to construct the *PWL_Benchmark Examples*. The left and right slopes of the fuzzy rule and observation play a significant role in preserving the conclusion's linearity. To represent the fuzzy sets of the antecedent, consequent, and observation in the *PWL_Benchmark Examples*, we use one-dimensional input and output variables, triangular membership function, and two fuzzy rules. *PWL_Benchmark Examples* and their results tested by the MATLAB FRI toolbox. The current version of the FRI toolbox is freely available to download in [24].

PWL_Benchmark Examples are divided into two groups. The first group presents the rule-base, observation configurations, where the KH-FRI conclusion is satisfied with the PWL condition. The second group shows the examples that the conclusions of the KH-FRI are not satisfied with the PWL condition. We will discuss the cases related to the PWL property of the KH-FRI conclusion.

The KH-FRI conclusion is always satisfied with PWL condition if the following cases are met:

For Case *PWLC1*: When the left and right slopes A_i and B_i fuzzy sets are identical *(e.g., for left slop* $a_{12} - a_{11} = a_{22} - a_{21}$ *and* $b_{12} - b_{11} = b_{22} - b_{21}$*)*, therefore, the conclusion of KH-FRI will always be satisfied with the linearity condition. **Table 21** illustrates the PWL_Notations of the Example "*PWL.LIN.C1*" that demonstrate the linearity conclusion related to ***Case PWLC1***.

Table 21. *The Preserving PWL Conclusion of The KH-FRI with Fuzzy Sets and PWL_Notations to Case PWLC1.*

<table>
<tr><td colspan="3">Example "PWL.LIN.C1"</td></tr>
<tr><td colspan="2">The characteristic points of the fuzzy sets:
A_1= [0 2 2 6] A_2= [10 12 12 16] A^*= [7 8 8 9]
B_1= [0 2 2 6] B_2= [10 12 12 16] B^*= [7 8 8 9]</td><td>The length of left and right slopes of the fuzzy sets:
For left: $A_1=2, A_2=2, A^*=1, B_1=2, B_2=2$
For Right: $A_1=4, A_2=4, A^*=1, B_1=4, B_2=4$</td></tr>
<tr><td>PWL_Notations by Eq.(6.32):
ΔB^* Left = 0
ΔB^* Right = 0</td><td>PWL_Notations by Eq.(6.33):
E.Left = NAN
E.Right = NAN</td><td>PWL_Notations by Eq.(6.34):
Left.Slope = 1
Right.Slope = 1</td></tr>
</table>

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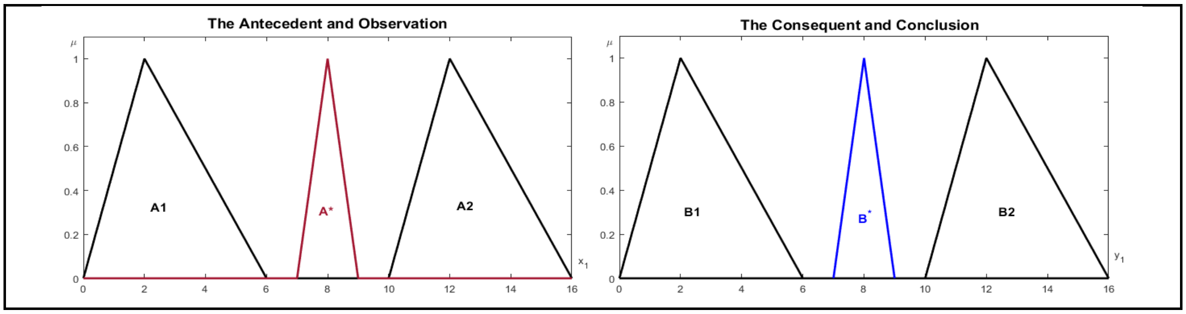
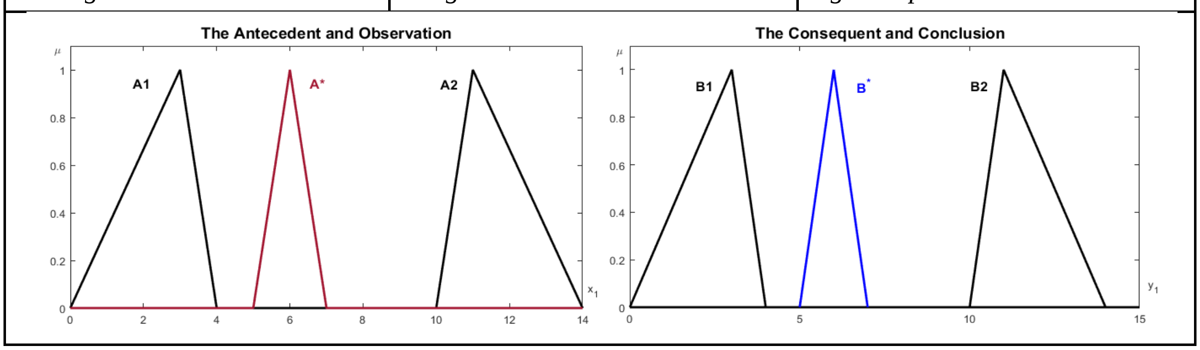


For Case *PWLC2*: If two adjacent fuzzy rule bases $A_1 \rightarrow B_1$ *and* $A_2 \rightarrow B_2$ *(e.g., for left slop: Rule1* $(a_{12} - a_{11} = b_{12} - b_{11})$*, Rule2* $(a_{22} - a_{21} = b_{22} - b_{21})$ have the same left and right slopes and the same characteristic points on the universe of discourse, then the KH-FRI conclusion will always be satisfied with the linearity condition. **Table 22** explains the Example "*PWL.LIN.C2*" that indicate to ***Case PWLC2***.

Table 22. *The Preserving PWL Conclusion of The KH-FRI with Fuzzy Sets and PWL_Notations to Case PWLC2.*

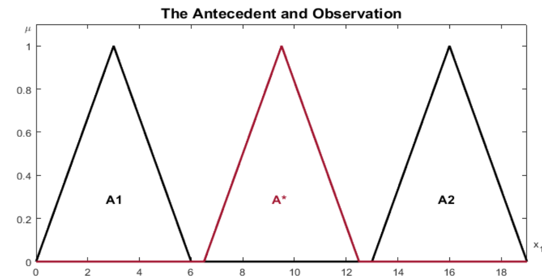
<table>
<tr><td colspan="3">Example "PWL.LIN.C2"</td></tr>
<tr><td colspan="2">The characteristic points of the fuzzy sets:
A_1= [0 3 3 4] A_2= [10 11 11 14] A^*= [5 6 6 7]
B_1= [0 3 3 4] B_2= [10 11 11 14] B^*= [5 6 6 7]</td><td>The length of left and right slopes of the fuzzy sets:
For left: $A_1=3, A_2=1, A^*=1, B_1=3, B_2=1$
For Right: $A_1=1, A_2=3, A^*=1, B_1=1, B_2=3$</td></tr>
<tr><td>PWL_Notations by Eq.(6.32):
ΔB^* Left = 0
ΔB^* Right= 0</td><td>PWL_Notations by Eq.(6.33):
E.Left = 0
E.Right = 0</td><td>PWL_Notations by Eq.(6.34):
Left.Slope = 1
Right.Slope = 1</td></tr>
</table>

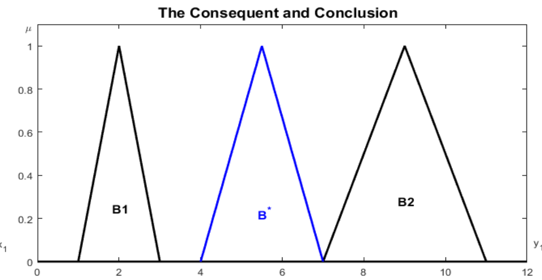


For Case *PWLC3*: When the fuzzy sets of the antecedents A_i and the observation A^* have the same left and right slopes PWL, therefore, the conclusion of the KH-FRI will always be satisfied with the linearity condition. **Table 23** defined PWL_Notations of Example "*PWL.LIN.C3*" regard to ***Case PWLC3***.

Table 23. *The Preserving PWL Conclusion of The KH-FRI with Fuzzy Sets and PWL_Notations to Case PWLC3.*

<table>
<tr><td colspan="3">Example "PWL.LIN.C3"</td></tr>
<tr><td colspan="2">The characteristic points of the fuzzy sets:
A_1= [0 3 3 6] A_2= [13 16 16 19] A^*= [6.5 9.5 9.5 12.5] B_1= [1 2 2 3] B_2= [7 9 9 11] B^*= [4 5.5 5.5 7]</td><td>The length of left and right slopes of the fuzzy sets:
For left: $A_1=3, A_2=3, A^*=3, B_1=1, B_2=2$
For Right: $A_1=3, A_2=3, A^*=3, B_1=1, B_2=2$</td></tr>
<tr><td>PWL_Notations by Eq.(6.32):
ΔB^* Left = 0
ΔB^* Right = 0</td><td>PWL_Notations by Eq.(6.33):
E.Left = NAN
E.Right = NAN</td><td>PWL_Notations by Eq.(6.34):
Left.Slope = 1
Right.Slope = 1</td></tr>
</table>

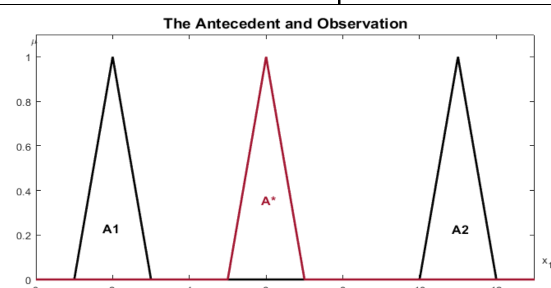


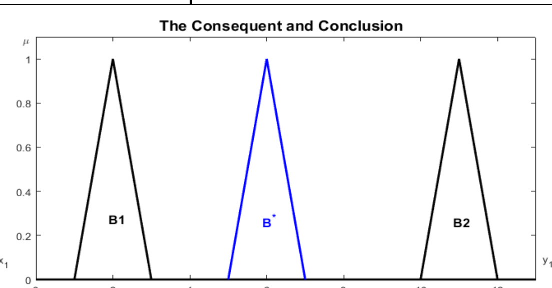


For Case *PWLC4*: When the left and right slopes for all fuzzy sets of two adjacent fuzzy rule bases and observation are equidistant *(A_i = B_i = A^*)*, therefore, the conclusion of the KH-FRI will always be satisfied with the linearity condition. **Table 24** illustrates PWL_Notations to Example "*PWL.LIN.C4*" which indicates to ***Case PWLC4***.

Table 24. *The Preserving PWL Conclusion of The KH-FRI with Fuzzy Sets and PWL_Notations to Case PWLC4.*

<table>
<tr><td colspan="3">Example "PWL.LIN.C4"</td></tr>
<tr><td colspan="2">The characteristic points of the fuzzy sets:
A_1= [1 2 2 3] A_2= [10 11 11 12] A^*= [5 6 6 7]
B_1= [1 2 2 3] B_2= [10 11 11 12] B^*= [5 6 6 7]</td><td>The length of left and right slopes of the fuzzy sets:
For left: $A_1=1, A_2=1, A^*=1, B_1=1, B_2=1$
For Right: $A_1=1\ A_2=1, A^*=1, B_1=1, B_2=1$</td></tr>
<tr><td>PWL_Notations by Eq.(6.32):
ΔB^* Left = 0
ΔB^* Right = 0</td><td>PWL_Notations by Eq.(6.33):
E.Left = NAN
E.Right = NAN</td><td>PWL_Notations by Eq.(6.34):
Left.Slope = 1
Right.Slope = 1</td></tr>
</table>



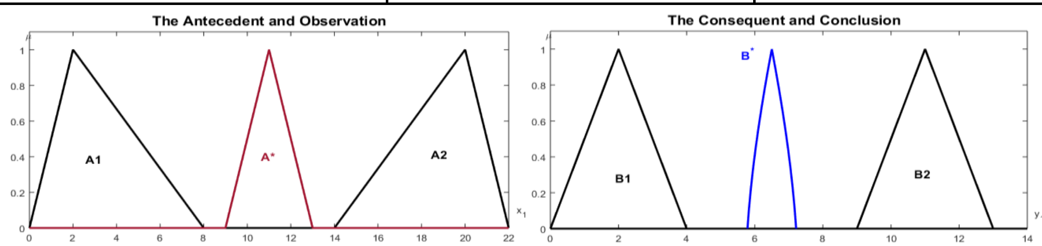
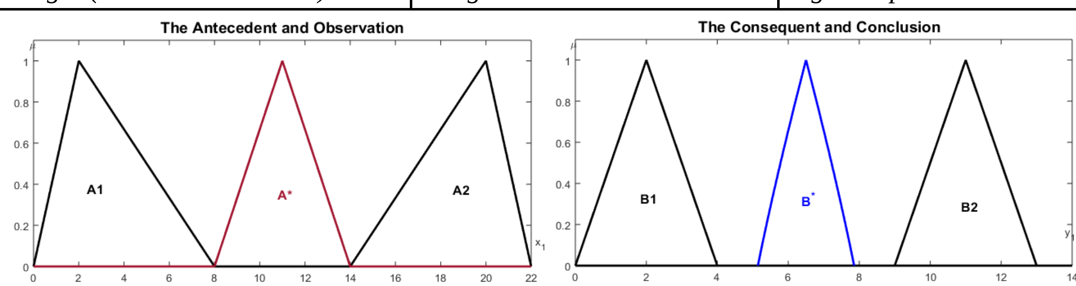


However, the conclusions of the KH-FRI are not satisfied with PWL condition based on **Eq.(6.32)**, **Eq.(6.33)** and **Eq.(6.34)** if the following cases hold.

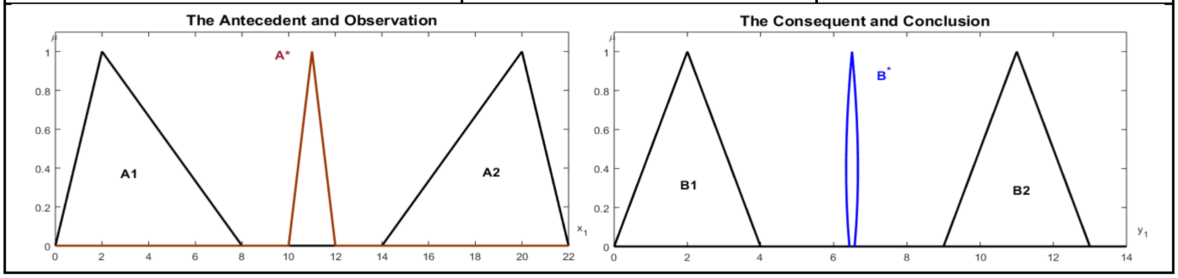
According to Case *PWLC1*: When the left and right slopes A_i and B_i are incompatible *(e.g., for left slop* $(a_{12} - a_{11} \neq a_{22} - a_{21})$ *and* $(b_{12} - b_{11} = b_{22} - b_{21})$, whereas $A_i \neq A^*$, in this case, the linearity conclusion of KH-FRI is not satisfied. Example "*PWL.NONLIN.C1*" constructed to prove the problem, which will be described by three different situations based on the characteristic points of the observation A^* to compare its linearity conclusions.

Table 25 illustrates PWL_Notations that describe the problem according to the three situations.

Table 25. *The Problem with Slopes to Case PWLC1 Which is not Preserving PWL*

<table>
<tr><td colspan="3">Example "PWL.NONLIN.C1" situation1 when e.g., the left slope (b1(2) - b1(1) = b2(2) - b2(1)) = A*</td></tr>
<tr><td>The characteristic points of the fuzzy sets:
A_1= [0 2 2 8] A_2= [14 20 20 22] A^*= [9 11 11 13]
B_1= [0 2 2 4] B_2= [9 11 11 13]
B^*= [5.79 6.50 6.50 7.21]</td><td colspan="2">The length of left and right slopes of the fuzzy sets:
For left: $A_1=2, A_2=6, A^*=2, B_1=2, B_2=2$
For Right: $A_1=6, A_2=2, A^*=2, B_1=2, B_2=2$</td></tr>
<tr><td>PWL_Notations by Eq.(6.32):
ΔB^* Left (Maximum-deviation) = 0.08
ΔB^* Right (Maximum-deviation) = 0.08</td><td>PWL_Notations by Eq.(6.33):
E.Left = 1.2857
E.Right = 1.2857</td><td>PWL_Notations by Eq.(6.34):
Left.Slope = 0
Right.Slope = 0</td></tr>
<tr><td colspan="3">
</td></tr>
<tr><td colspan="3">Example "PWL.NONLIN.C1" situation2 when e.g., the left slope (b1(2) - b1(1) = b2(2) - b2(1)) < A*</td></tr>
<tr><td>The characteristic points of the fuzzy sets:
A_1= [0 2 2 8] A_2= [14 20 20 22] A^*= [8 11 11 14]
B_1= [0 2 2 4] B_2= [9 11 11 13]
B^*= [5.14 6.50 6.50 7.86]</td><td colspan="2">The length of left and right slopes of the fuzzy sets:
For left: $A_1=2, A_2=6, A^*=3, B_1=2, B_2=2$
For Right: $A_1=6, A_2=2, A^*=3, B_1=2, B_2=2$</td></tr>
<tr><td>PWL_Notations by Eq.(6.32):
ΔB^* Left (Maximum-deviation) = 0.04
ΔB^* Right (Maximum-deviation) = 0.04</td><td>PWL_Notations by Eq.(6.33):
E.Left = 0.6429
E.Right = 0.6429</td><td>PWL_Notations by Eq.(6.34):
Left.Slope = 0
Right.Slope = 0</td></tr>
<tr><td colspan="3">
</td></tr>
<tr><td colspan="3">Example "PWL.NONLIN.C1" situation3 when e.g., the left slope (b1(2) - b1(1) = b2(2) - b2(1)) > A*</td></tr>
<tr><td>The characteristic points of the fuzzy sets:
A_1= [0 2 2 8] A_2= [14 20 20 22] A^*= [10 11 11 12]
B_1= [0 2 2 4] B_2= [9 11 11 13]
B^*= [6.4286 6.5 6.5 6.571]</td><td colspan="2">The length of left and right slopes of the fuzzy sets:
For left: $A_1=2, A_2=6, A^*=1, B_1=2, B_2=2$
For Right: $A_1=6, A_2=2, A^*=1, B_1=2, B_2=2$</td></tr>
</table>

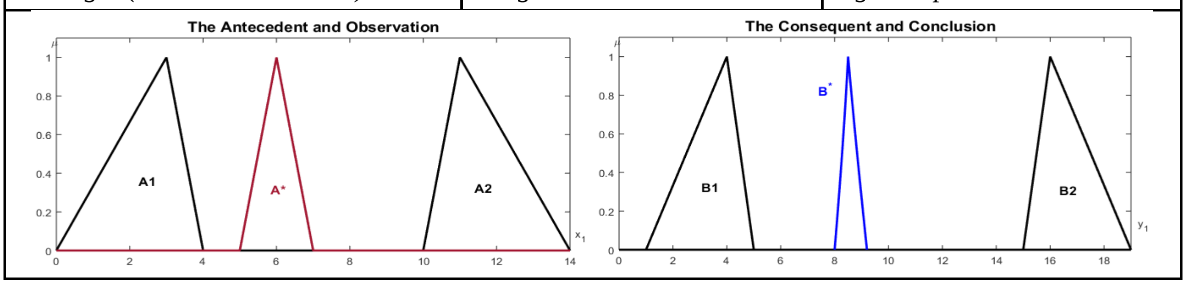
PWL_Notations by Eq.(6.32):	PWL_Notations by Eq.(6.33):	PWL_Notations by Eq.(6.34):
ΔB^* *Left (Maximum-deviation) = 0.121*	*E.Left = 1.9286*	*Left.Slope = 0*
ΔB^* *Right (Maximum-deviation) = 0.121*	*E.Right = 1.9286*	*Right.Slope = 0*



About Case *PWLC2*: When the two adjacent fuzzy rule bases $A_1 \rightarrow B_1$ and $A_2 \rightarrow B_2$ have the same left and right slopes but have different characteristic points on the universe of discourse, in this case, the linearity conclusion of KH-FRI is not satisfied. Example "*PWL.NONLIN.C2*" constructed to prove the issue, as shown in **Table 26**.

Table 26. *The Problem with Slopes to Case PWLC2 Which is not Preserving PWL*

Example "*PWL.NONLIN.C2*"		
The characteristic points of the fuzzy sets: A_1= [0 3 3 4] A_2= [10 11 11 14] A^*= [5 6 6 7] B_1= [1 4 4 5] B_2= [15 16 16 19] B^*= [8 8.5 8.5 9.2]	**The length of left and right slopes of the fuzzy sets:** ***For left***: $A_1=3, A_2=1, A^*=1, B_1=3, B_2=1$ ***For Right***: $A_1=1, A_2=3, A^*=1, B_1=1, B_2=3$	
PWL_Notations by Eq.(6.32): ΔB^* *Left (Maximum-deviation) = 0.028* ΔB^* *Right (Maximum-deviation) = 0.017*	**PWL_Notations by Eq.(6.33):** *E.Left = 0.500* *E.Right = 0.300*	**PWL_Notations by Eq.(6.34):** *Left.Slope = 0* *Right.Slope = 0*



Referring to Case *PWLC3*: When the left and right slopes of the antecedents A_i $(a_{12} - a_{11} = a_{22} - a_{21})$ and the observation A^* are not equivalent, whereas $A_i \neq B_i$, then the linearity conclusion of KH-FRI is not satisfied. Refer to Corollary *PWLCR1*, Example "*PWL.NONLIN.C3*" is applied the polynomial condition when $(a_{12} - a_{11} = a_{22} - a_{21})$; however, it is not linear. **Table 27** describes PWL_Notations which prove the problem to this case.

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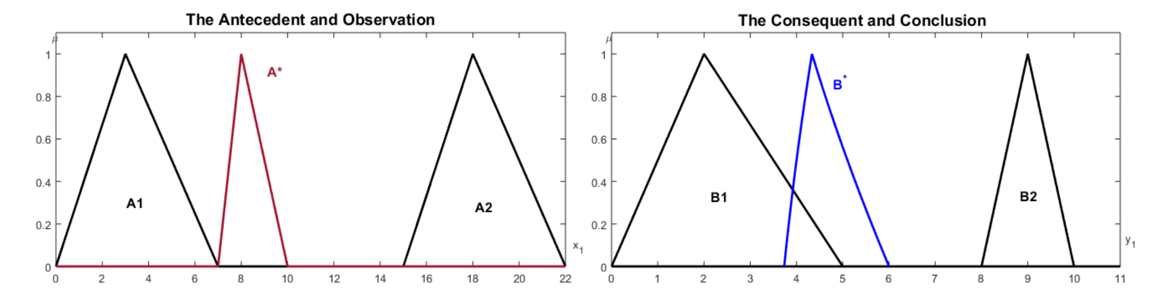
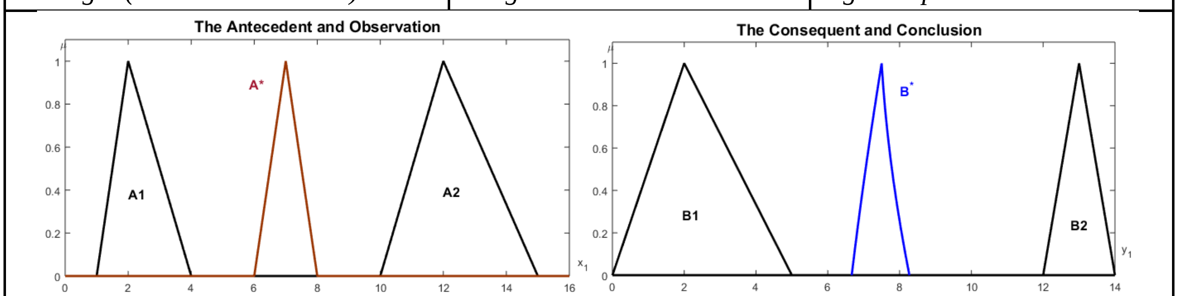
Table 27. *The Problem with Slopes to Case PWLC3 Which is not Preserving PWL*

<table>
<tr><td colspan="3">Example "PWL.NONLIN.C3"</td></tr>
<tr><td colspan="2">The characteristic points of the fuzzy sets:
A_1= [0 3 3 7] A_2= [15 18 18 22] A^*= [7 8 8 10]
B_1= [0 2 2 5] B_2= [8 9 9 10] B^*= [3.73 4.33 4.33 6.0]</td><td>The length of left and right slopes of the fuzzy sets:
For left: $A_1=3, A_2=3, A^*=1, B_1=2, B_2=1$
For Right: $A_1=4, A_2=4, A^*=2, B_1=3, B_2=1$</td></tr>
<tr><td>PWL_Notations by Eq.(6.32):
ΔB^* Left (Maximum-deviation) = 0.033
ΔB^* Right (Maximum-deviation) = 0.067</td><td>PWL_Notations by Eq.(6.33):
E.Left = NAN
E.Right = NAN</td><td>PWL_Notations by Eq.(6.34):
Left.Slope = 0
Right.Slope = 0</td></tr>
<tr><td colspan="3"></td></tr>
</table>

According to Case *PWLC4*: When values of the left and right slopes of fuzzy rule bases and observation are not similar $(A_i \neq B_i \neq A^*)$, then the linearity conclusion of KH-FRI is not satisfied. Example "*PWL.NONLIN.C4*" created to demonstrate the problem, as shown in **Table 28**.

Table 28. *The Problem with Slopes to Case PWLC4 Which is not Preserving PWL*

<table>
<tr><td colspan="3">Example "PWL.NONLIN.C4"</td></tr>
<tr><td colspan="2">The characteristic points of the fuzzy sets:
A_1= [1 2 2 4] A_2= [10 12 12 15] A^*= [6 7 7 8]
B_1= [0 2 2 5] B_2= [12 13 13 14]
B^*= [6.67 7.5 7.5 8.27]</td><td>The length of left and right slopes of the fuzzy sets:
For left: $A_1=1, A_2=2, A^*=1, B_1=2, B_2=1$
For Right: $A_1=2, A_2=3, A^*=1, B_1=3, B_2=1$</td></tr>
<tr><td>PWL_Notations by Eq.(6.32):
ΔB^*Left (Maximum-deviation) = 0.031
ΔB^*Right (Maximum-deviation) = 0.101</td><td>PWL_Notations by Eq.(6.33):
E.Left = 1.1667
E.Right = 4.2273</td><td>PWL_Notations by Eq.(6.34):
Left.Slope = 0
Right.Slope = 0</td></tr>
<tr><td colspan="3"></td></tr>
</table>

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6.3.5. Discussion of the KH-FRI PWL benchmark examples

In the following, the *PWL_Benchmark Examples* and their notations will be discussed in detail. Examples "*PWL.LIN.C1*" to "*PWL.LIN.C4*" shown in **Table 21 - Table 24** demonstrate PWL configurations, where the conclusions of the KH-FRI are always preserving *PWL* property, according to **Eq.(6.32)** is always equal to *0* because the values of the real and linear approximation functions are similar. Also, by **Eq.(6.33)** is *NAN* or, in some cases, is equal to *0* because the parameters cL_9 or cR_9 are equal to *0* (see Corollary *PWLCR1* and *PWLCR2*). **Eq.(6.34)** could always be satisfied with preserving the *PWL* when left.Slope and right.Slope are *1*.

According to examples of the first group, two examples will be taken to demonstrate the linearity *(PWL)* of the KH-FRI.

Referring to Example *"PWL.LIN.C1"* as shown in Table 21, we conclude the following:

- The conclusion of the KH-FRI is satisfied with PWL condition related to **Case *PWLC1***, where, the support length of the left and right slopes of antecedent A_i and consequent *Bi* fuzzy sets are similar.
- Using **Eq.(6.32)**, **Eq.(6.33)**, and **Eq.(6.34),** the conclusion is always satisfied with the linearity property. E.g.,
  - The left slope of antecedent fuzzy sets: ($A_1=2$ *and* $A_2=2$) and ($B_1=2$ *and* $B_2=2$).
  - The right slope of antecedent fuzzy sets: ($A_1=4$ *and* $A_2=4$) and ($B_1=4$ *and* $B_2=4$).
- **Fig. 80** shows the result of ΔB^*, which is obtained by **Eq.(6.32)** for all α-cut levels to the left and right slopes that are equal to *0*.

α - Levels		0.000	0.100	0.200	0.300	0.400	0.500	0.600	0.700	0.800	0.900	1.000
Left Slope	Real B*	5.000	5.100	5.200	5.300	5.400	5.500	5.600	5.700	5.800	5.900	6.000
	approx B*	5.000	5.100	5.200	5.300	5.400	5.500	5.600	5.700	5.800	5.900	6.000
	ΔB*	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
Right Slope	Real B*	6.000	6.100	6.200	6.300	6.400	6.500	6.600	6.700	6.800	6.900	7.000
	approx B*	6.000	6.100	6.200	6.300	6.400	6.500	6.600	6.700	6.800	6.900	7.000
	ΔB*	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Fig. 80. *The Difference Between the Linear Approximation and Real Functions of the Left and Right Slopes for α ∈ [0,1] To Example "PWL.LIN.C1"*

- The estimated error PWL_Notations are proved the non-linearity as follows:
  - By **Eq.(6.33)** is *NAN* for left and right slopes (*E.left = NAN*, *E.right = NAN*), the PWL_Notations of Corollary *PWLCR1* and *PWLCR2* are also demonstrated, where (e.g., for left slope) DL_1 is *0* (because of $cL_9 = 0$).
  - By **Eq.(6.34)** is *1* for left and right slopes (*Left.Slope = 1*, *Right.Slope = 1*).(6.33)

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Another case of the preserving linearity, it is evident by Example "*PWL.LIN.C2*" as shown in Table 22, we conclude the following:

- The conclusion of the KH-FRI is satisfied with linearity condition when slopes of two fuzzy rule bases are equivalent; this example is restricted to the characteristic points of the two fuzzy rule bases, which must be identified as mentioned in **Case *PWLC2***. E.g.,
  - The Rule $A_1 \Rightarrow B_1$ (for the lower $A_1=3$ *and* $B_1=3$), and (for the upper $A_1=1$ *and* $B_1=1$).
  - The Rule $A_2 \Rightarrow B_2$ (for the lower $A_1=1$ *and* $B_1=1$), and (for the upper $A_1=3$ *and* $B_1=3$).
  - The characteristic points $A_1= [0\ 3\ 3\ 4] \Rightarrow B_1= [0\ 3\ 3\ 4]$ *and* $A_2= [10\ 11\ 11\ 14] \Rightarrow B_2= [10\ 11\ 11\ 14]$.
- ΔB^* of the left and right slopes that are equal *0*, which computed by **Eq.(6.32)**.
- The estimated error PWL_Notations are proved the non-linearity as follows:
  - By **Eq.(6.33)** is *0* for left and right slopes (*E.left = 0*, *E.right = 0*), despite the parameters cL_9 and DL_1 (by PWL_Notations of **Eq.(6.34)**) are "*not zero*", e.g., for left slope, $cL_9 = -2$ and *DL1* are *-2*.
  - By **Eq.(6.34)** is *1* for left and right slopes (*Left.Slope = 1*, *Right.Slope = 1*).(6.33)

In contrast, Examples *"PWL.NONLIN.C1" – "PWL.NONLIN.C4"* describe cases, in which the conclusions of the KH-FRI are not satisfied with PWL (the second group). These examples have been presented based on two facts, either if the conclusion is close to linearity (Example "*PWL.NONLIN.C1*" situation2) or far from linearity (Example "*PWL.NONLIN.C1*" situation3).

According to **Eq.(6.34),** the conclusion is not satisfied with the PWL condition because the values are always *0* for (*left.Slope* and *right.Slope*). **Eq.(6.32)** and **Eq.(6.33)** will be discussed in detail as follows:

According to Example "*PWL.NONLIN.C1*" as shown in Table 25, we conclude the following:

- There are three different situations based on the characteristic points of the consequents and the observation as:
  - Situation1 *(when* $(b_{12} - b_{11} = b_{22} - b_{21}) = A^*$*)*,
  - Situation2 (*when* $(b_{12} - b_{11} = b_{22} - b_{21}) < A^*$*)*,
  - Situation3 *(when* $(b_{12} - b_{11} = b_{22} - b_{21}) > A^*$).
- **Fig. 81** shows the difference between real and linear approximation functions for each situation. Using **Eq.(6.32)**, the maximum deviation for left and right slopes in situation1 is *0.08*, and situation2 is smaller than situation1, which is *0.04*, in contrast, situation3 has the high deviation is *0.121*.

α - Levels		0.000	0.100	0.200	0.300	0.400	0.500	0.600	0.700	0.800	0.900	1.000
ΔB* Left Slope	Situation 1	0.000	0.032	0.056	0.071	0.079	0.080	0.075	0.064	0.048	0.026	0.000
	Situation 2	0.000	0.016	0.028	0.036	0.040	0.040	0.038	0.032	0.024	0.013	0.000
	Situation 3	0.000	0.048	0.083	0.107	0.119	0.121	0.113	0.096	0.072	0.039	0.000
ΔB* Right Slope	Situation 1	0.000	0.026	0.048	0.064	0.075	0.080	0.079	0.071	0.056	0.032	0.000
	Situation 2	0.000	0.013	0.024	0.032	0.038	0.040	0.040	0.036	0.028	0.016	0.000
	Situation 3	0.000	0.039	0.072	0.096	0.113	0.121	0.119	0.107	0.083	0.048	0.000

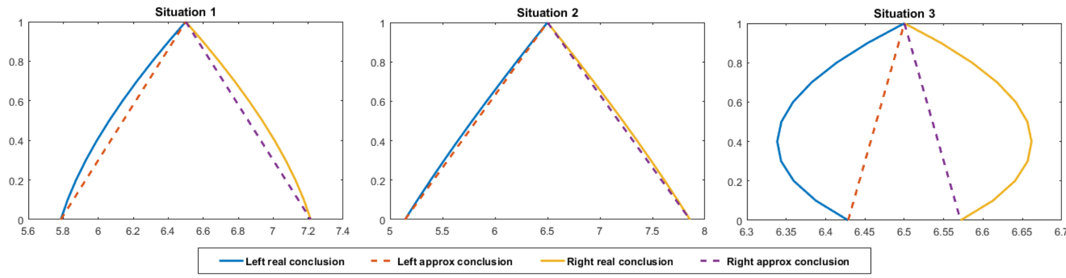

Fig. 81. *The Difference Between the Linear Approximation and Real Functions of the Left and Right Slopes for α ∈ [0,1] To Example "PWL.NONLIN.C1"*

- On the other hand, **Eq.(6.33)** shows the error ratios for the three situations, situation3 has a large error ratio compared to situation1 and situation2, where the error ratio of the left and right slopes of (situation3 is 1.9286), (situation1 is *1.2857)*, and (situation2 is *0.6429)*. Therefore, situation3 is far from linearity, in contrast to situation2, which is closer than situation1 to linearity.
- Using **Eq.(6.34)**, the conclusion is always not satisfied with the PWL condition, where the value is always 0 for (left.Slope and right.Slope).

In Example "*PWL.NONLIN.C2*" as shown in Table 26, we conclude the following:

- The problem appears when the left and right slopes of fuzzy rule bases are the same, but the characteristic points of the fuzzy sets of A_i and B_i are different on the universe of discourse. In this case, the conclusion KH-FRI is not satisfied with linearity. E.g.,
  - The Rule $A_1 \Rightarrow B_1$ (for the lower $A_1=3$ *and* $B_1=3$), and (for the upper $A_1=1$ *and* $B_1=1$).
  - The Rule $A_2 \Rightarrow B_2$ (for the lower $A_1=1$ *and* $B_1=1$), and (for the upper $A_1=3$ *and* $B_1=3$).
  - The characteristic points $A_1= [0\ 3\ 3\ 4] \Rightarrow B_1= [1\ 4\ 4\ 5]$ *and* $A_2= [10\ 11\ 11\ 14] \Rightarrow B_2= [15\ 16\ 16\ 19]$.
- Referring to **Eq.(6.32)**, the deviation for the left slope is greater than the right slope, where the left slope is *0.028*, and the right slope is *0.017*, as shown in **Fig. 82**.

α - Levels	ΔB*	
	Left Slope	Right Slope
0.000	0.000	0.000
0.100	0.009	0.007
0.200	0.017	0.011
0.300	0.022	0.015
0.400	0.026	0.016
0.500	0.028	0.017
0.600	0.027	0.016
0.700	0.024	0.013
0.800	0.019	0.010
0.900	0.011	0.006
1.000	0.000	0.000

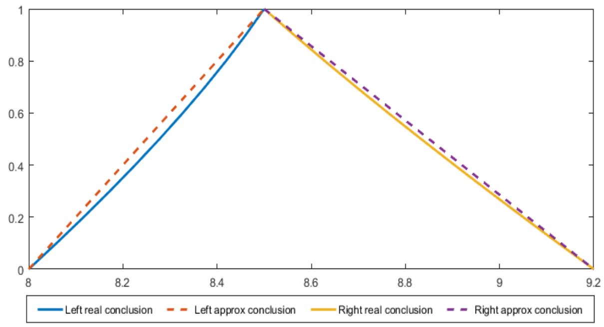

Fig. 82. *The Difference Between the Linear Approximation and Real Functions of the Left and Right Slopes for α ∈ [0,1] To Example "PWL.NONLIN.C2"*

- Also, **Eq.(6.33)** describes the error ratio, where the left slope is *0.5* is far from linearity to the right slope is *0.3*.
- According to **Eq.(6.34)**, the conclusion is always not satisfied with the PWL condition, where the value is always 0 for (left.Slope and right.Slope).

In Example "*PWL.NONLIN.C3*" as shown in Table 27, we conclude the following:

- The conclusion of KH-FRI is not satisfied with linearity according to the results of equations and PWL_Notations, when the left and right slopes of the antecedents A_i $(a_{12} - a_{11} = a_{22} - a_{21})$ and the observation A^* are not equivalent whereas $A_i \neq B_i$.
- **Fig. 83** shows the difference between real and linear approximation functions, where the maximum deviation for left and right slopes are *0.033* and *0.067*, respectively, by **Eq.(6.32)**.

α - Levels	ΔB*	
	Left Slope	Right Slope
0.000	0.000	0.000
0.100	0.012	0.024
0.200	0.021	0.043
0.300	0.028	0.056
0.400	0.032	0.064
0.500	0.033	0.067
0.600	0.032	0.064
0.700	0.028	0.056
0.800	0.021	0.043
0.900	0.012	0.024
1.000	0.000	0.000

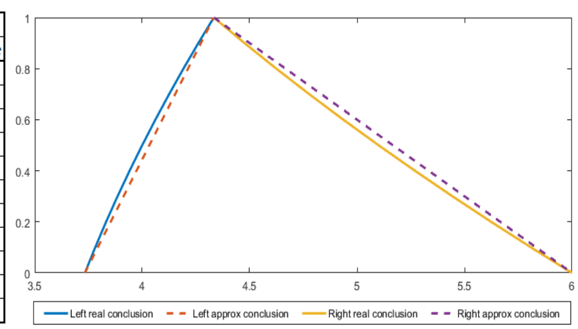

Fig. 83. *The Difference Between the Linear Approximation and Real Functions of the Left and Right Slopes for α ∈ [0,1] To Example "PWL.NONLIN.C3"*

- **Eq.(6.33)** determines the error ratio for the left and the right slopes are *NAN*, by referring to Corollary *PWLCR1*, this example has achieved the condition of polynomiality because the left and right slopes of A_1 and A_2 are similar, but not linear.
- Using **Eq.(6.34)**, the conclusion is always not satisfied with the PWL condition because the value of this equation is always 0 for (left.Slope and right.Slope).

Also, in Example "*PWL.NONLIN.C4*" as shown in Table 28, we conclude the following:

- All fuzzy sets of fuzzy rule bases and observation are different *($A_i \neq B_i \neq A^*$)*, then the conclusion KH-FRI is also not satisfied with the linearity condition.
- Additionally, **Fig. 84** shows the difference between real and linear approximation functions as follows:

α - Levels	ΔB*	
	Left Slope	Right Slope
0.000	0.000	0.000
0.100	0.409	0.207
0.200	0.535	0.365
0.300	0.556	0.475
0.400	0.525	0.538
0.500	0.467	0.556
0.600	0.390	0.529
0.700	0.303	0.459
0.800	0.207	0.347
0.900	0.106	0.193
1.000	0.000	0.000

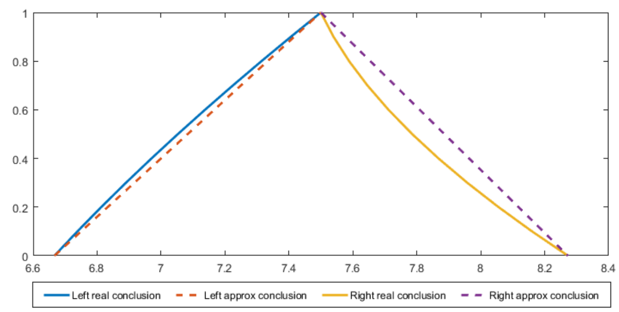

Fig. 84. *The Difference Between the Linear Approximation and Real Functions of the Left and Right Slopes for α ∈ [0,1] To Example "PWL.NONLIN.C4"*

- **Eq.(6.33)** determines the error ratio of the linearity in the right slope is *4.2273*, which is so far from the left slope of which is *1.1667*.
- According to **Eq.(6.34)**, the conclusion is always not satisfied with the PWL condition because the value is always 0 for (left.Slope and right.Slope).

Based on all examples and cases of the PWL benchmark, we conclude the following:

1. All examples of the first group of the *PWL_Benchmark Examples* are satisfied with PWL property, as proved by the difference between *Left and Right slopes* condition between the fuzzy rules and observation, the conclusion is always preserving on the linearity if one of the below conditions is met:
   - The left and right slopes A_i and B_i fuzzy sets are identical.
   - The two adjacent fuzzy rule bases $A_1 \rightarrow B_1$ and $A_2 \rightarrow B_2$ have the same left and right slopes and the same characteristic points on the universe of discourse.
   - The fuzzy sets of the antecedents A_i and the observation A^* have the same left and right slopes.
   - The left and right slopes for all fuzzy sets of two adjacent fuzzy rule bases and observation are equidistant ($A_i = B_i = A^*$).

Moreover, all PWL_Notations of the Left and Right slopes (The error of approximating the nonlinear slope **Eq.(6.32)**, the upper limit of the error **Eq.(6.33)**, the linearity error **Eq.(6.34)** of the fuzzy sets and the difference between real function and linear approximation) are also satisfied with the PWL condition.

2. On the other hand, all examples of the second group of the *PWL_Benchmark Examples* are not satisfied with PWL property, as proved by the difference between *Left and Right slopes*

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condition of the fuzzy rules and observation, where the conclusion is always not preserving on the linearity if one of the below conditions is met:

- The left and right slopes A_i and B_i are incompatible, whereas $A_i \neq A^*$, and (Situation1: $B_i = A^*$, Situation2: $B_i < A^*$, Situation3: $B_i > A^*$).
- The two adjacent fuzzy rule bases $A_1 \rightarrow B_1$ and $A_2 \rightarrow B_2$ have the same left and right slopes but have different characteristic points on the universe of discourse.
- The left and right slopes of the antecedents A_i $(a_{12} - a_{11} = a_{22} - a_{21})$ and the observation A^* are not equivalent, whereas $A_i \neq B_i$.
- The values of the left and right slopes of fuzzy rule bases and observation are not similar ($A_i \neq B_i \neq A^*$).

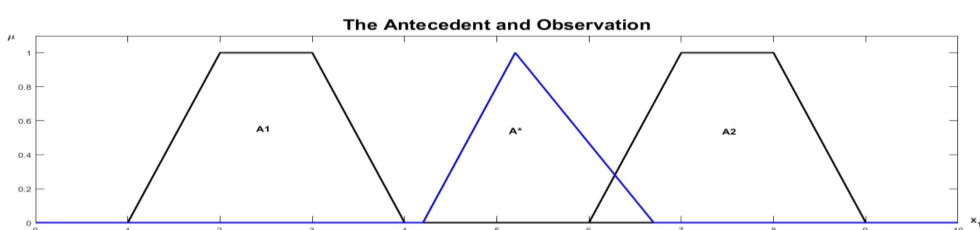
Moreover, all PWL_Notations of the Left and Right slopes (**Eq.(6.32)**, **Eq.(6.33)**, **Eq.(6.34),** and the difference between real function and linear approximation) proved that the conclusion is always not satisfied with the PWL condition.

6.4. The Application of the CNF and PWL Benchmark Examples

This subsection will discuss the efficiency of the CNF and PWL benchmarks compared to some of the FRI methods implemented via the FRI Toolbox. Besides, the validity of the proposed Incircle-FRI method will be discussed based on this benchmark and compared to other FRI methods.

6.4.1. Testing FRI methods based on the CNF benchmark examples

In the following, some of the FRI methods (KHstabilized [5], MACI [6], VKK [4], and CRF [7]) will be tested and compared according to the *CNF_Benchmark Examples*. To offer a simple way of comparison, we focus on the cases that demonstrated abnormal conclusion (Examples: "*CNF.KH_ABNOR.C4_1*" – "*CNF.KH_ABNOR.C4_4*"). This comparison shows the difference between the results of the selected methods according to the CNF property. **Fig. 85** introduces the antecedents and observation part of Examples ("*CNF.KH_ABNOR.C4_1*" – "*CNF.KH_ABNOR.C4_4*") as shown in **Table 17** – **Table 20** and **Fig. 86** - **Fig. 89** describe the results of the FRI methods (KHstabilized [5], MACI [6], VKK [4], and CRF [7]).



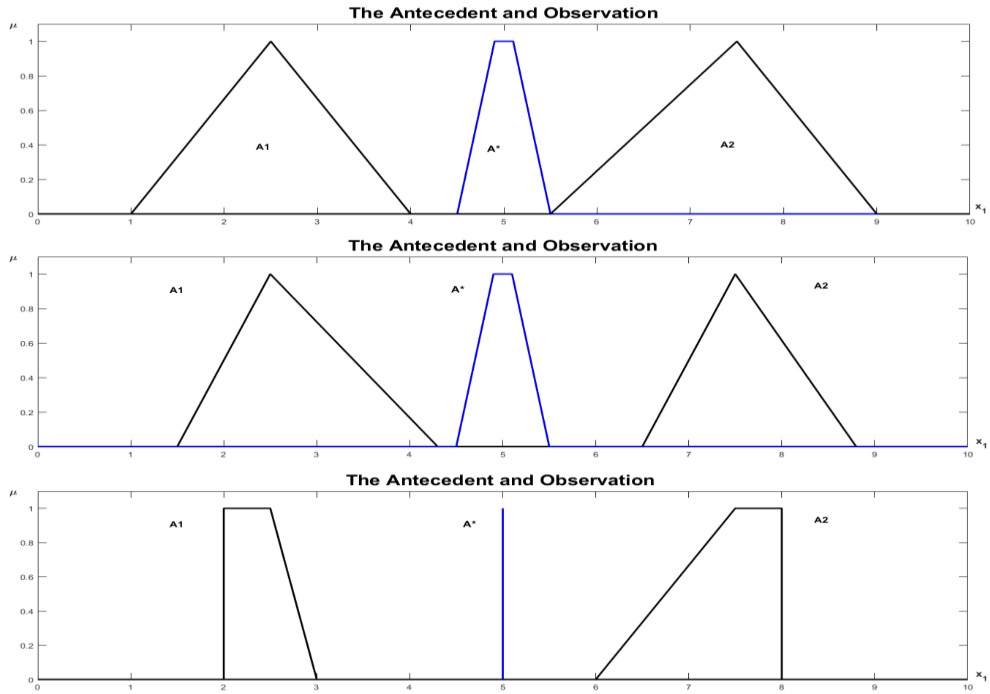
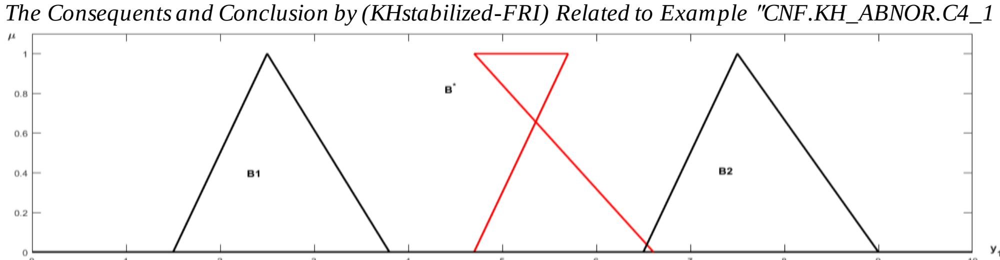
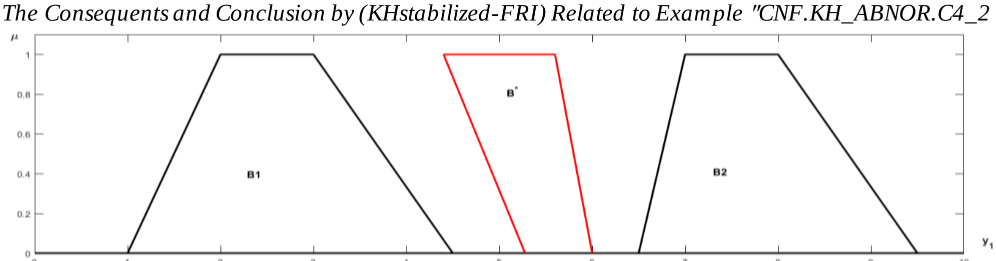


Fig. 85. *The Antecedents and Observations Related to Examples ("CNF.KH_ABNOR.C4_1"–"CNF.KH_ABNOR.C4_4")*

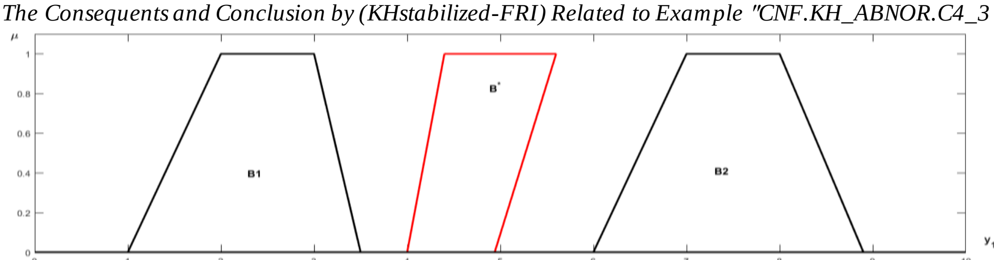
The Consequents and Conclusion by (KHstabilized-FRI) Related to Example "CNF.KH_ABNOR.C4_1"



The Consequents and Conclusion by (KHstabilized-FRI) Related to Example "CNF.KH_ABNOR.C4_2"



The Consequents and Conclusion by (KHstabilized-FRI) Related to Example "CNF.KH_ABNOR.C4_3"



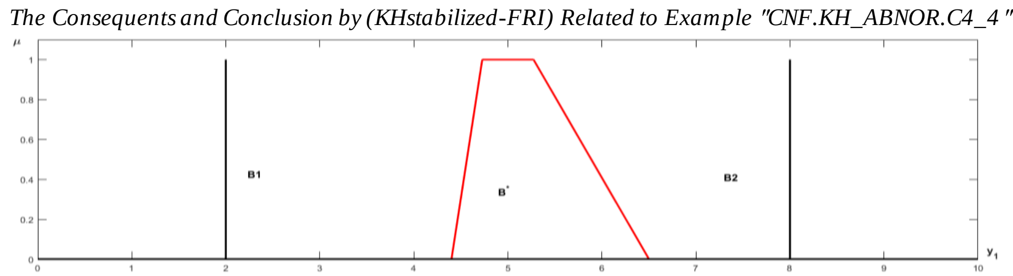


Fig. 86. *The Conclusions of the KHstabilized-FRI Method Related to CNF_Benchmark Examples ("CNF.KH_ABNOR.C4_1"– "CNF.KH_ABNOR.C4_4")*

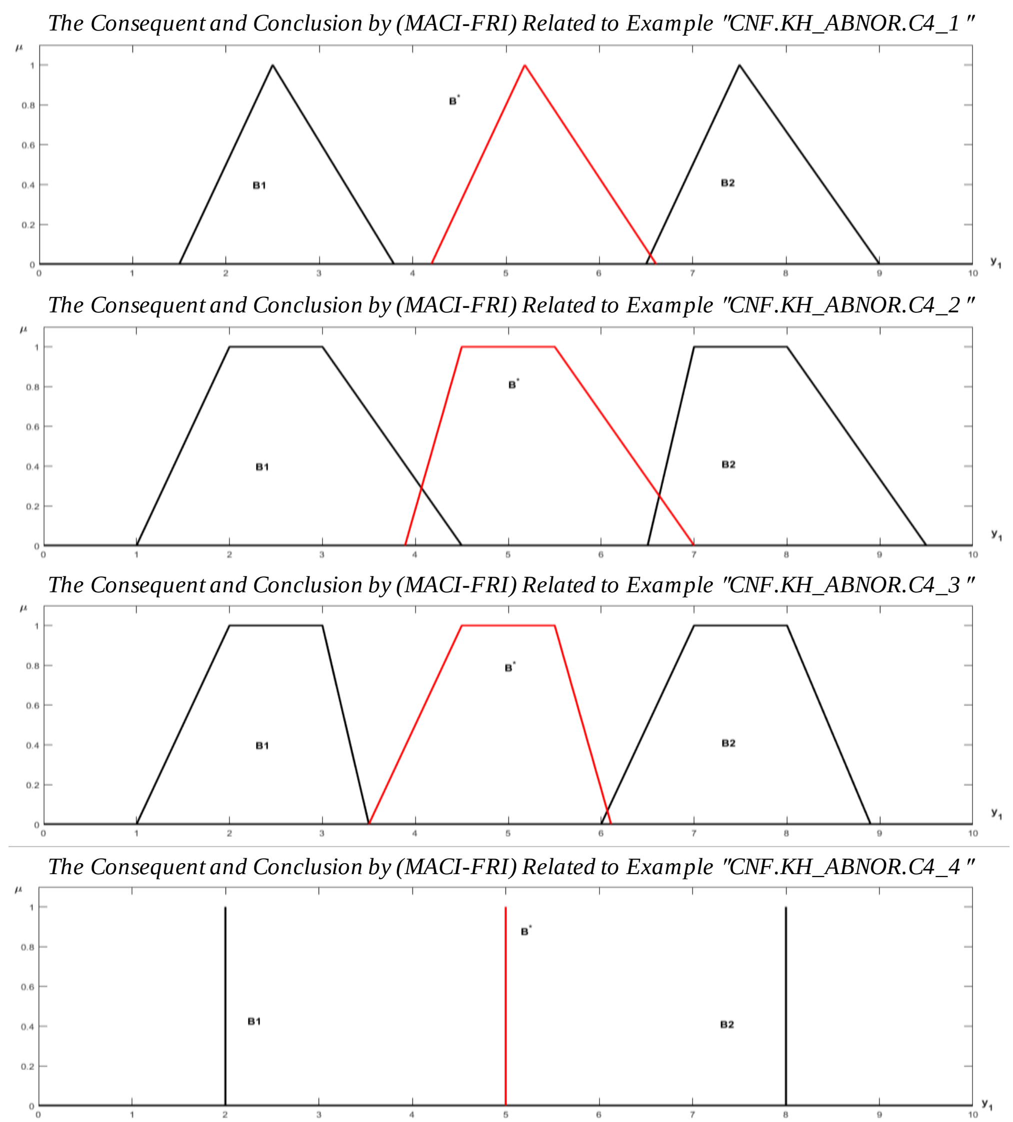


Fig. 87. *The Conclusions of the MACI-FRI Method Related to CNF_Benchmark Examples ("CNF.KH_ABNOR.C4_1"– "CNF.KH_ABNOR.C4_4")*

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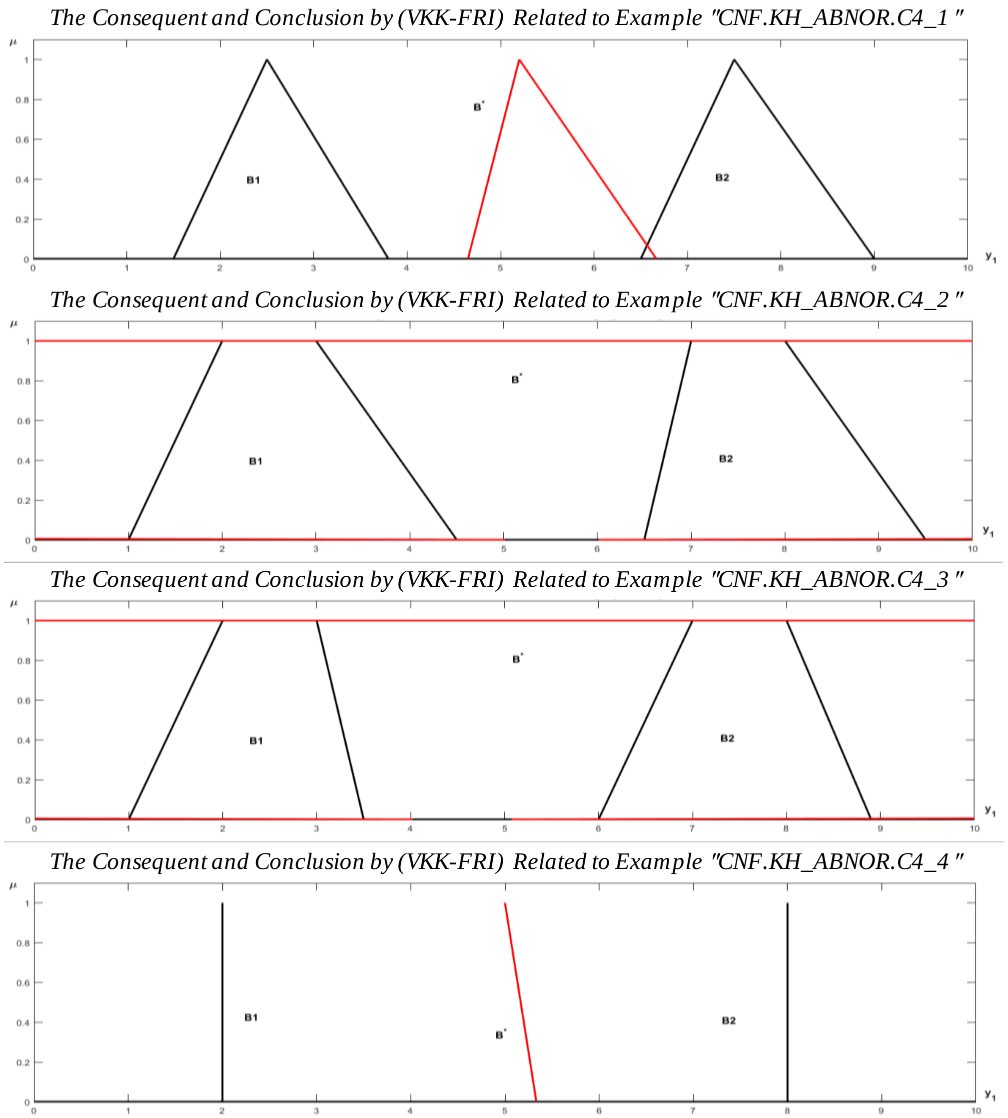
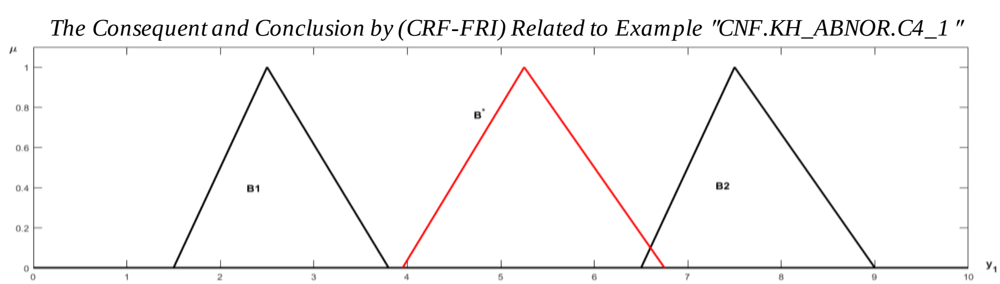


Fig. 88. *The Conclusions of the VKK-FRI Method Related to CNF_Benchmark Examples ("CNF.KH_ABNOR.C4_1" – "CNF.KH_ABNOR.C4_4")*



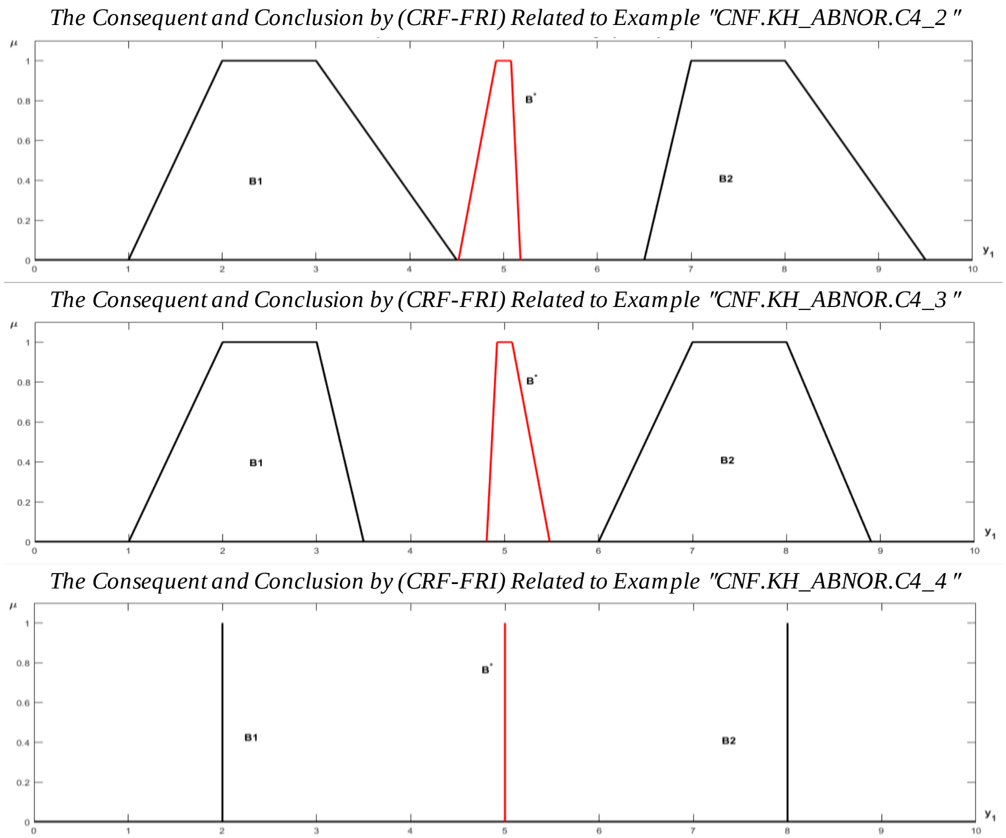


Fig. 89. *The Conclusions of the CRF-FRI Method Related to CNF_Benchmark Examples ("CNF.KH_ABNOR.C4_1 "– "CNF.KH_ABNOR.C4_4 ")*

6.4.2. Testing the Incircle-FRI method based on the CNF benchmark examples

In the following, the proposed Incircle-FRI will be tested to prove the validity of the conclusions according to the CNF property. First, the Incircle-FRI will be tested according to *CNF_Benchmark Examples* (see section 6.2). Second, it will be tested according to all examples used in chapter 4. Third, it will be tested using examples with different core and fuzziness sides between fuzzy rules and observation fuzzy sets.

1. According to the *CNF_Benchmark Examples* presented in subsection 6.2, which was constructed to test the CNF property, in the following, we examine the Incircle-FRI based on this benchmark. **Table 29 - Table 32** introduce the CNF_Notations that prove the preservation of the CNF property of the Incircle-FRI.

Table 29. *The CNF_Notations of the Incircle-FRI Result Related to Example "CNF.KH_ABNOR.C4_1" (in subsection 6.2), Which is Preserving CNF*

The values of the fuzzy sets: A_1= [1 2 3 4] A_2= [6 7 8 9] B_1= [1.5 2.5 2.5 3.8] B_2= [6.5 7.5 7.5 9] A^*= [4.2 5.2 5.2 6.7] B^*= [4.82 5.55 5.55 6.49]	**The case of the Core, LFs and RFs conclusion:** The length (LFs) is (**NORMAL**) The length (Core) is (**NORMAL**) The length (RFs) is (**NORMAL**)
The lengths of the Core and Fuzziness sides: LFS_length: *LFS1 = (0), LFS2 = (0.88)* CoreL: *Core1 = (-56.7273), Core2= (431.0204)* CoreR: *Core1 = (-25.06), Core2= (566.41)* RFS_length: *RFS1= (0.0941), RFS2= (2.423)*	**Length ratios of the Core and Fuzziness sides:** *LFS.Ratio1 = (0), LFS.Ratio2 = (0)* *CoreL.Ratio1 = (1), CoreL.Ratio2 = (1)* *CoreR.Ratio1 = (1), CoreR.Ratio2 = (1)* *RFS.Ratio1 = (0), RFS.Ratio2 = (0)*

The Consequent and Conclusion by (Incircle-FRI) Related to Example "CNF.KH_ABNOR.C4_1"

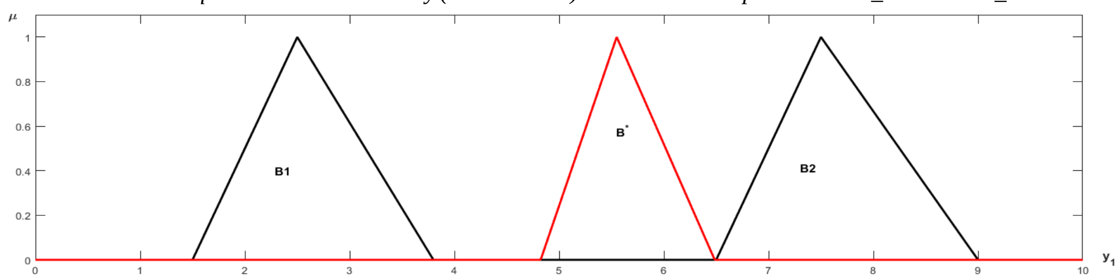


Table 30. *The CNF_Notations of the Incircle-FRI Result Related to Example "CNF.KH_ABNOR.C4_2" (in subsection 6.2), Which is Preserving CNF*

The values of the fuzzy sets: A_1= [1 2.5 2.5 4] A_2= [5.5 7.5 7.5 9] B_1= [1 2 3 4.5] B_2= [6.5 7 8 9.5] A^*= [4.5 4.9 5.1 5.5] B^*= [4.36 4.86 5.29 5.87]	**The case of the Core, LFs and RFs conclusion:** The length (LFs) is (NORMAL) The length (Core) is (NORMAL) The length (RFs) is (NORMAL)
The lengths of the Core and Fuzziness sides: LFS_length: *LFS1 = (-0.71), LFS2 = (4.20)* CoreL: *Core1 = (-34.0508), Core2 = (383.2101)* CoreR: *Core1 = (-28.722), Core2= (567.0)* RFS_length: *RFS1= (0), RFS2= (0.767)*	**Length ratios of the Core and Fuzziness sides:** *LFS.Ratio1 = (0), LFS.Ratio2 = (1)* *CoreL.Ratio1 = (1), CoreL.Ratio2 = (1)* *CoreR.Ratio1 = (1), CoreR.Ratio2 = (1)* *RFS.Ratio1 = (0), RFS.Ratio2 = (0)*

The Consequent and Conclusion by (Incircle-FRI) Related to Example "CNF.KH_ABNOR.C4_2"

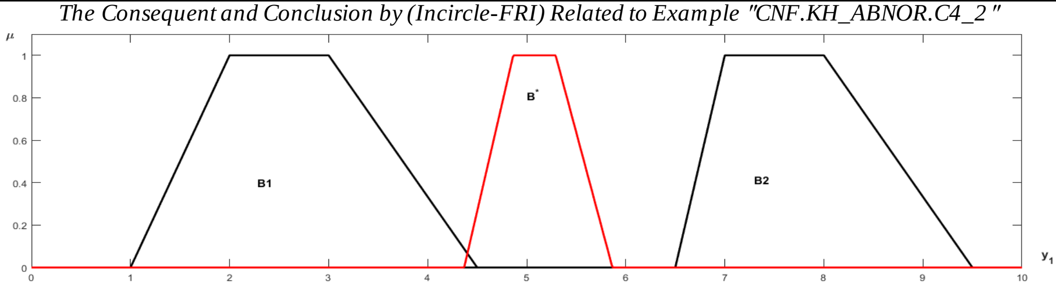


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Table 31. *The CNF_Notations of the Incircle-FRI Result Related to Example "CNF.KH_ABNOR.C4_3" (in subsection 6.2), Which is Preserving CNF*

The values of the fuzzy sets: A_1= [1.5 2.5 2.5 4.3] A_2= [6.5 7.5 7.5 8.8] B_1= [1 2 3 3.5] B_2= [6 7 8 8.9] A^*= [4.5 4.9 5.1 5.5] B^*= [4.10 4.80 5.23 5.65]	**The case of the Core, LFs and RFs conclusion:** The length (LFs) is (NORMAL) The length (Core) is (NORMAL) The length (RFs) is (NORMAL)
The lengths of the Core and Fuzziness sides: LFS_length: *LFS1 = (0), LFS2 = (0.5137)* CoreL: *Core1 = (-32.58), Core2= (381.728)* CoreR: *Core1 = (-28.89), Core2= (558.87)* RFS_length: *RFS1= (0.409), RFS2= (0.681)*	**Length ratios of the Core and Fuzziness sides:** *LFS.Ratio1 = (0), LFS.Ratio2 = (0)* *CoreL.Ratio1 = (1), CoreL.Ratio2 = (1)* *CoreR.Ratio1 = (1), CoreR.Ratio2 = (1)* *RFS.Ratio1 = (-0.68), RFS.Ratio2 = (1)*

The Consequent and Conclusion by (Incircle-FRI) Related to Example "CNF.KH_ABNOR.C4_3"

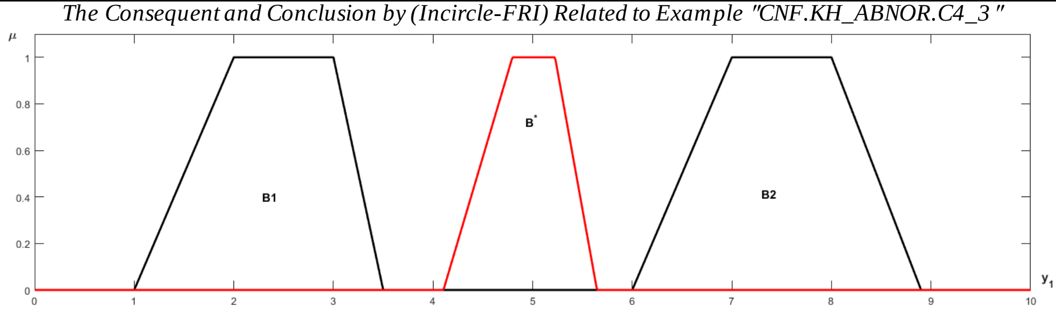
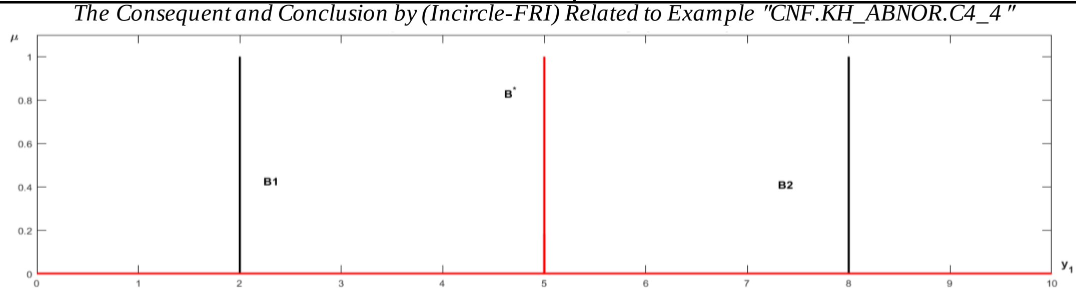


Table 32. *The CNF_Notations of the Incircle-FRI Result Related to Example "CNF.KH_ABNOR.C4_4" (in subsection 6.2), Which is Preserving CNF*

The values of the fuzzy sets: A_1= [2 2 2.5 3] A_2= [6 7.5 8 8] B_1= [2 2 2 2] B_2= [8 8 8 8] A^*= [5 5 5 5] B^*= [5.03 5.023 5.03 5.03]	**The case of the Core, LFs and RFs conclusion:** The length (LFs) is (NORMAL) The length (Core) is (NORMAL) The length (RFs) is (NORMAL)
The lengths of the Core and Fuzziness sides: LFS_length: *LFS1 = (0), LFS2 = (0)* CoreL: *Core1 = (-57.2133), Core2 = (458.7421)* CoreR: *Core1 = (-28.48), Core2= (467.93)* RFS_length: *RFS1= (0), RFS2= (0)*	**Length ratios of the Core and Fuzziness sides:** *LFS.Ratio1 = (0), LFS.Ratio2 = (1)* *CoreL.Ratio1 = (1), CoreL.Ratio2 = (1)* *CoreR.Ratio1 = (1), CoreR.Ratio2 = (1)* *RFS.Ratio1 = (0), RFS.Ratio2 = (1)*

The Consequent and Conclusion by (Incircle-FRI) Related to Example "CNF.KH_ABNOR.C4_4"



2. According to the suggested examples in chapter 4, which was presented to test the proposed Incircle-FRI, in the following. **Table 33 - Table 38** introduce the CNF_Notations that demonstrate the CNF property of the Incircle-FRI.

Table 33. *The CNF_Notations of the Incircle-FRI Result Related to Example CNFIncircle(TR1) (in chapter 4), Which is Preserving CNF*

Example *CNFIncircle(TR1)* In case the (A_i), (A^*), and (B_i) are Triangular	
The values of the fuzzy sets: A_1= [0 5 6] A_2= [11 13 14] B_1= [0 2 4] B_2= [10 11 13] A^*= [7 8 9] B^*= [4.95 5.42 7.16]	**The case of the Core, LFs and RFs conclusion:** The length (LFs) is (NORMAL) The length (Core) is (NORMAL) The length (RFs) is (NORMAL)
The lengths of the Core and Fuzziness sides: LFS_length: *LFS1 = (-6.25), LFS2 = (5.05)* CoreL: *Core1 = (0.104), Core2= (1.29)* CoreR: *Core1 = (13.07), Core2= (15.83)* RFS_length: *RFS1= (0), RFS2= (2.288)*	**Length ratios of the Core and Fuzziness sides:** *LFS.Ratio1 = (0.310), LFS.Ratio2 = (1)* *CoreL.Ratio1 = (1), CoreL.Ratio2 = (1)* *CoreR.Ratio1 = (1), CoreR.Ratio2 = (1)* *RFS.Ratio1 = (0), RFS.Ratio2 = (0)*

Table 34. *The CNF_Notations of the Incircle-FRI Result Related to Example CNFIncircle(TR2) (in chapter 4), Which is Preserving CNF*

Example *CNFIncircle(TR2)* In case the (A_i) and (B_i) are Triangular, (A^*) is Singlton	
The values of the fuzzy sets: A_1= [0 5 6] A_2= [11 13 14] B_1= [0 2 4] B_2= [10 11 13] A^*= [8 8 8] B^*= [5.42 5.42 5.42]	**The case of the Core, LFs and RFs conclusion:** The length (LFs) is (NORMAL) The length (Core) is (NORMAL) The length (RFs) is (NORMAL)
The lengths of the Core and Fuzziness sides: LFS_length: *LFS1 = (-6.705), LFS2 = (8.472)* CoreL: *Core1 = (-32.15), Core2= (1359)* CoreR: *Core1 = (46.16), Core2= (1520)* RFS_length: *RFS1= (0), RFS2= (1.14)*	**Length ratios of the Core and Fuzziness sides:** *LFS.Ratio1 = (0.310), LFS.Ratio2 = (1)* *CoreL.Ratio1 = (1), CoreL.Ratio2 = (1)* *CoreR.Ratio1 = (1), CoreR.Ratio2 = (1)* *RFS.Ratio1 = (0), RFS.Ratio2 = (0)*

Table 35. *The CNF_Notations of the Incircle-FRI Result Related to Example CNFIncircle(TR3) (in chapter 4), Which is Preserving CNF*

Example *CNFIncircle(TR3)* In case the (A_i) is Singlton, and (A^*) is Triangular and (B_i) (Singlton & Triangular)	
The values of the fuzzy sets: A_1= [3 3 3] A_2= [12 12 12] B_1= [4 4 4] B_2= [10 11 13] A^*= [5 6 8] B^*= [5.28 6.37 8.28]	**The case of the Core, LFs and RFs conclusion:** The length (LFs) is (NORMAL) The length (Core) is (NORMAL) The length (RFs) is (NORMAL)
The lengths of the Core and Fuzziness sides: LFS_length: *LFS1 = (-0.353), LFS2 = (0.707)* CoreL: *Core1 = (18.06), Core2= (1406)* CoreR: *Core1 = (-22.63), Core2= (1521)* RFS_length: *RFS1= (-4.236), RFS2= (8.472)*	**Length ratios of the Core and Fuzziness sides:** *LFS.Ratio1 = (0), LFS.Ratio2 = (0)* *CoreL.Ratio1 = (0.75), CoreL.Ratio2 = (1)* *CoreR.Ratio1 = (0.80), CoreR.Ratio2 = (1)* *RFS.Ratio1 = (0), RFS.Ratio2 = (0)*

Table 36. *The CNF_Notations of the Incircle-FRI Result Related to Example CNFIncircle(TP1) (in chapter 4), Which is Preserving CNF*

Example *CNFIncircle(TP1)* In case the (A_i) and (B_i) are Trapezoidal, and (A^*) is Triangular	
The values of the fuzzy sets: A_1= [0 4 5 6] A_2= [11 12 13 14] B_1= [0 2 3 4] B_2= [10 11 12 13] A^*= [6 6 9 10] B^*= [4.54 4.54 7.47 8.53]	**The case of the Core, LFs and RFs conclusion:** The length (LFs) is (NORMAL) The length (Core) is (NORMAL) The length (RFs) is (NORMAL)
The lengths of the Core and Fuzziness sides: **LFS_length**: *LFS1 = (-3.069), LFS2 = (2.918)* **CoreL**: *Core1 = (68.90), Core2= (989.37)* **CoreR**: *Core1 = (-130.9), Core2= (2081)* **RFS_length**: *RFS1= (0), RFS2= (1.896)*	**Length ratios of the Core and Fuzziness sides:** *LFS.Ratio1 = (0.319), LFS.Ratio2 = (1)* *CoreL.Ratio1 = (1), CoreL.Ratio2 = (1)* *CoreR.Ratio1 = (1), CoreR.Ratio2 = (1)* *RFS.Ratio1 = (0), RFS.Ratio2 = (0)*

Table 37. *The CNF_Notations of the Incircle-FRI Result Related to Example CNFIncircle(TP2) (in chapter 4), Which is Preserving CNF*

Example *CNFIncircle(TP2)* In case the (A_i) and (A^*) are Triangular, and (B_i) is Trapezoidal	
The values of the fuzzy sets: A_1= [0 5 6] A_2= [11 13 14] B_1= [0 2 3 4] B_2= [10 11 12 13] A^*= [7 8 9] B^*= [5.01 5.90 5.90 7.33]	**The case of the Core, LFs and RFs conclusion:** The length (LFs) is (NORMAL) The length (Core) is (NORMAL) The length (RFs) is (NORMAL)
The lengths of the Core and Fuzziness sides: LFS_length: *LFS1 = (-6.250), LFS2 = (5.4113)* CoreL: *Core1 = (0.104), Core2= (135)* CoreR: *Core1 = (13.0), Core2= (1851)* RFS_length: *RFS1= (0), RFS2= (1.270)*	**Length ratios of the Core and Fuzziness sides:** *LFS.Ratio1 = (0.310), LFS.Ratio2 = (1)* *CoreL.Ratio1 = (1), CoreL.Ratio2 = (1)* *CoreR.Ratio1 = (1), CoreR.Ratio2 = (1)* *RFS.Ratio1 = (0), RFS.Ratio2 = (0)*

Table 38. *The CNF_Notations of the Incircle-FRI Result Related to Example CNFIncircle(TP3) (in chapter 4), Which is Preserving CNF*

Example *CNFIncircle(TP3)* In case (A_i) is Trapezoidal, and (A^*) and (B_i) are Triangular	
The values of the fuzzy sets: A_1= [0 4 5 6] A_2= [11 12 13 14] B_1= [0 2 4] B_2= [10 11 13] A^*= [6 6 9 10] B^*= [4.37 4.37 6.66 8.57]	**The case of the Core, LFs and RFs conclusion:** The length (LFs) is (NORMAL) The length (Core) is (NORMAL) The length (RFs) is (NORMAL)
The lengths of the Core and Fuzziness sides: LFS_length: *LFS1 = (-3.069), LFS2 = (2.610)* CoreL: *Core1 = (68.90), Core2= (945.35)* CoreR: *Core1 = (-130.9), Core2= (180)* RFS_length: *RFS1= (0), RFS2= (3.416)*	**Length ratios of the Core and Fuzziness sides:** *LFS.Ratio1 = (0.310), LFS.Ratio2 = (1)* *CoreL.Ratio1 = (1), CoreL.Ratio2 = (1)* *CoreR.Ratio1 = (1), CoreR.Ratio2 = (1)* *RFS.Ratio1 = (0), RFS.Ratio2 = (0)*

3. The following examples present with different core and fuzziness sides of the fuzzy sets to examine the CNF property of the Incircle-FRI. **Table 39** - **Table 42** describe the CNF_Notations that prove the preservation of the CNF property of the Incircle-FRI.

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***Table 39.** The CNF_Notations of the Incircle-FRI Result Related to Example CNFIncircle(Diff1), Which is Preserving CNF*

Example *CNFIncircle(Diff1)* In case (A_i) and (B_i) are Singlton, and (A^*) is Triangular	
The values of the fuzzy sets: A_1= [2 2 2 2] A_2= [8 8 8 8] B_1= [3 3 3 3] B_2= [9 9 9 9] A^*= [4 5 5 6] B^*= [5.07 5.77 5.77 6.93]	**The case of the Core, LFs and RFs conclusion:** The length (LFs) is (*NORMAL*) The length (Core) is (*NORMAL*) The length (RFs) is (*NORMAL*)
The lengths of the Core and Fuzziness sides: LFS_length: *LFS1 = (0), LFS2 = (0)* CoreL: *Core1 = (-40.71), Core2= (593.66)* CoreR: *Core1 = (-679), Core2= (656.18)* RFS_length: *RFS1= (0), RFS2= (0)*	**Length ratios of the Core and Fuzziness sides:** *LFS.Ratio1 = (0), LFS.Ratio2 = (0)* *CoreL.Ratio1 = (1), CoreL.Ratio2 = (1)* *CoreR.Ratio1 = (1), CoreR.Ratio2 = (1)* *RFS.Ratio1 = (0), RFS.Ratio2 = (0)*

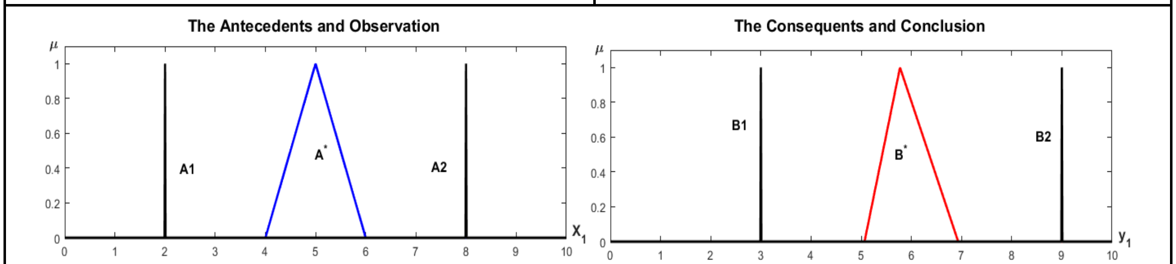

***Table 40.** The CNF_Notations of the Incircle-FRI Result Related to Example CNFIncircle(Diff2), Which is Preserving CNF*

Example *CNFIncircle(Diff2)* In case (A_i), (A^*), and (B_i) are Triangular fuzzy sets (A^* is Compatible with Rule1)	
The values of the fuzzy sets: A_1= [1 2 2 3] A_2= [9 10 10 11] B_1= [2 3 3 4] B_2= [9 11 11 13] A^*= [1 2 2 3] B^*= [2.07 3.0 3.00 3.93]	**The case of the Core, LFs and RFs conclusion:** The length (LFs) is (*NORMAL*) The length (Core) is (*NORMAL*) The length (RFs) is (*NORMAL*)
The lengths of the Core and Fuzziness sides: LFS_length: *LFS1 = (0), LFS2 = (1.162)* CoreL: *Core1 = (113.1), Core2= (515.56)* CoreR: *Core1 = (142.9), Core2= (657.2)* RFS_length: *RFS1= (0), RFS2= (1.162)*	**Length ratios of the Core and Fuzziness sides:** *LFS.Ratio1 = (0), LFS.Ratio2 = (0)* *CoreL.Ratio1 = (1), CoreL.Ratio2 = (1)* *CoreR.Ratio1 = (1), CoreR.Ratio2 = (1)* *RFS.Ratio1 = (0), RFS.Ratio2 = (0)*

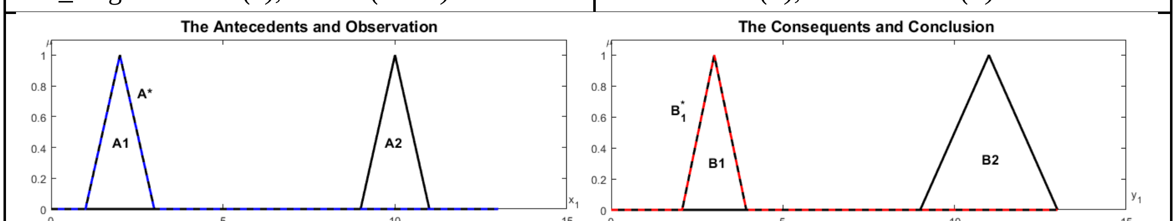

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Table 41. *The CNF_Notations of the Incircle-FRI Result Related to Example CNFIncircle(Diff3), Which is Preserving CNF*

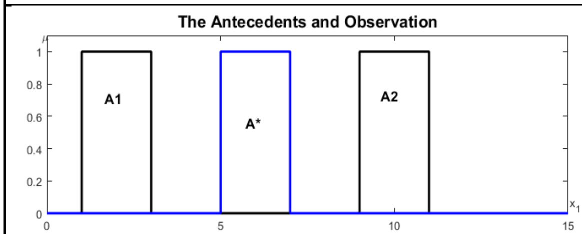
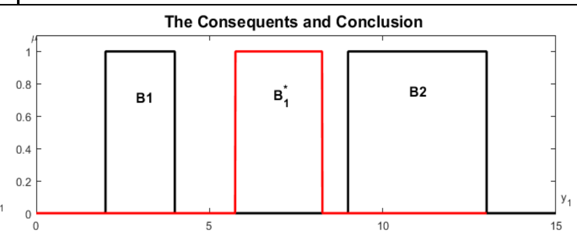
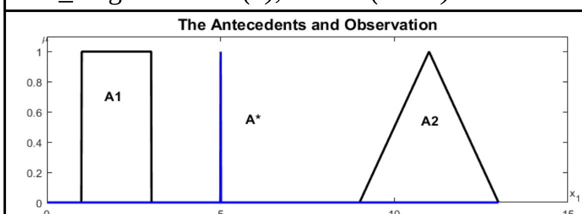
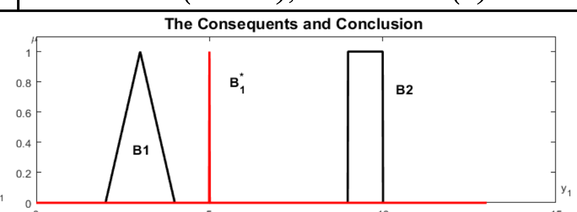
Example *CNFIncircle(Diff3)* In case (A_i), (A^*), and (B_i) are Trapezoidal fuzzy sets (when the left and right sides are equal zero)	
The values of the fuzzy sets: A_1= [1 1 3 3] A_2= [9 9 11 11] B_1= [2 2 4 4] B_2= [9 9 13 13] A^* = [5 5 7 7] B^*= [5.74 5.74 8.26 8.26]	**The case of the Core, LFs and RFs conclusion:** The length (LFs) is (*NORMAL*) The length (Core) is (*NORMAL*) The length (RFs) is (*NORMAL*)
The lengths of the Core and Fuzziness sides: LFS_length: *LFS1 = (0), LFS2 = (0)* CoreL: *Core1 = (-112.4), Core2= (718.0)* CoreR: *Core1 = (-143.5), Core2= (1.530)* RFS_length: *RFS1= (0), RFS2= (0)*	**Length ratios of the Core and Fuzziness sides:** *LFS.Ratio1 = (0), LFS.Ratio2 = (0)* *CoreL.Ratio1 = (0.849), CoreL.Ratio2 = (1)* *CoreR.Ratio1 = (1), CoreR.Ratio2 = (1)* *RFS.Ratio1 = (0), RFS.Ratio2 = (0)*
	

Table 42. *The CNF_Notations of the Incircle-FRI Result Related to Example CNFIncircle(Diff4), Which is Preserving CNF*

Example *CNFIncircle(Diff4)* In case (A_i), (A^*), and (B_i) are different core and fuzziness sides fuzzy sets	
The values of the fuzzy sets: A_1= [1 1 3 3] A_2= [9 11 11 13] B_1= [2 3 3 4] B_2= [9 9 10 10] A^*= [5 5 5 5] B^*= [5.13 5.13 5.13 5.13]	**The case of the Core, LFs and RFs conclusion:** The length (LFs) is (*NORMAL*) The length (Core) is (*NORMAL*) The length (RFs) is (*NORMAL*)
The lengths of the Core and Fuzziness sides: LFS_length: *LFS1 = (0), LFS2 = (3.70)* CoreL: *Core1 = (-75.57), Core2= (893.56)* CoreR: *Core1 = (41.05), Core2= (991.12)* RFS_length: *RFS1= (0), RFS2= (3.702)*	**Length ratios of the Core and Fuzziness sides:** *LFS.Ratio1 = (-0.437), LFS.Ratio2 = (1)* *CoreL.Ratio1 = (0.671), CoreL.Ratio2 = (1)* *CoreR.Ratio1 = (0.779), CoreR.Ratio2 = (1)* *RFS.Ratio1 = (-0.4370), RFS.Ratio2 = (1)*
	

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6.4.3. The CNF benchmark examples discussion

The CNF conclusion property of the studied FRI methods (KH-FRI [3], [25], [39], KHstabilized-FRI [5], MACI-FRI [6], VKK-FRI [4], and CRF-FRI [7]) as shown **in Fig. 86**, **Fig. 87**, **Fig. 88** and **Fig. 89**, concerning the *CNF_Benchmark Examples* "*CNF.KH_ABNOR.C4_1*" – "*CNF.KH_ABNOR.C4_4*" can be summarized as follows:

1. MACI-FRI and CRF-FRI methods are suitable approach to be implemented as an inference system because its conclusions succeeded with CNF property to *CNF_Benchmark Examples*.
2. VKK-FRI method, the abnormal conclusion exceeded in *CNF_Benchmark Example* "*CNF.KH_ABNOR.C4_1*" only. However, it failed with CNF property in *CNF_Benchmark Examples* "*CNF.KH_ABNOR.C4_2*", "*CNF.KH_ABNOR.C4_3*", and "*CNF.KH_ABNOR.C4_4*".
3. KHstabilized-FRI method suffered from abnormality according to *CNF_Benchmark Examples* "*CNF.KH_ABNOR.C4_1*" – "*CNF.KH_ABNOR.C4_4*".

Table 43 illustrates the results of the studied FRI methods, as shown by the values of the conclusions B^*.

Table 43. *The Conclusions of the Selected Methods (KH, KHstabilized, MACI, VKK, and CRF) Related to Examples "CNF.KH_ABNOR.C4_1" – "CNF.KH_ABNOR.C4_4"*

Method	Approximate Conclusion B^*			
	Example *CNF.KH_ABNOR.C4_1*	Example *CNF.KH_ABNOR.C4_2*	Example *CNF.KH_ABNOR.C4_3*	Example *CNF.KH_ABNOR.C4_4*
KH-FRI [3],[25],[39]	Abnormal conclusion [4.7 5.7 4.7 6.6]	Abnormal conclusion [5.27 4.4 5.6 6.0]	Abnormal conclusion [4 4.4 5.6 4.94]	Abnormal conclusion [6.5 5.27 4.72 4.4]
KHstabilized-FRI [5]	Abnormal conclusion [4.7 5.7 4.7 6.6]	Abnormal conclusion [5.27 4.4 5.6 6.0]	Abnormal conclusion [4 4.4 5.6 4.94]	Abnormal conclusion [6.5 5.27 4.72 4.4]
MACI-FRI [6]	Normal conclusion [4.2 5.2 5.2 6.6]	Normal conclusion [3.89 4.5 5 5 5.5 7]	Normal conclusion [3.5 4.5 5 5 5.5 6.1]	Normal conclusion [5 5 5 5]
VKK-FRI [4]	Normal conclusion [4.6 5.2 5.2 6.66]	Abnormal conclusion [out range]	Abnormal conclusion [out range]	Abnormal conclusion [5.3 5 5 5.3]
CRF-FRI [7]	Normal conclusion [3.9 5.25 5.25 6.75]	Normal conclusion [4.5 4.9 5.0 5.1]	Normal conclusion [4.8 4.9 5.0 5.4]	Normal conclusion [5 5 5 5]

According to *CNF_Benchmark Examples* ("*CNF.KH_ABNOR.C4_1*" – "*CNF.KH_ABNOR.C4_4*") and different Examples ("*CNFIncircle(TR1)*" – "*CNFIncircle(Diff4)*") and their results as shown in **Table 29** - **Table 32**, the Incircle-FRI conclusion is a suitable approach to be implemented as an inference system because its conclusions succeeded with CNF property with all examples. The validity of the Incircle-FRI method with CNF property, which is proved by values of the conclusions B^* as shown in **Table 44** and **Table 45**.

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Table 44. *The Conclusions of the Incircle-FRI Method Related to Examples ("CNF.KH_ABNOR.C4_1 "– "CNF.KH_ABNOR.C4_4 ")*

Approximate Conclusion B^* of the Incircle-FRI method			
Example *"CNF.KH_ABNOR.C4_1"*	Example *"CNF.KH_ABNOR.C4_2 "*	Example *"CNF.KH_ABNOR.C4_3 "*	Example *"CNF.KH_ABNOR.C4_4 "*
Normal [4.82 5.55 5.55 6.49]	Normal [4.36 4.86 5.29 5.87]	Normal [4.10 4.80 5.23 5.65]	Normal [5.03 5.023 5.03 5.03]

Table 45. *The Conclusions of the Incircle-FRI Method Related to Examples CNFIncircle(TR1) – CNFIncircle(Diff4)*

<table>
<tr><td colspan="12">Approximate Conclusion B^* of the Incircle-FRI method</td></tr>
<tr><td colspan="4">Example "CNFIncircle(TR1)"</td><td colspan="4">Example "CNFIncircle(TR2)"</td><td colspan="4">Example "CNFIncircle(TR3)"</td></tr>
<tr><td colspan="4">Normal
[4.95 5.42 7.16]</td><td colspan="4">Normal
[5.42 5.42 5.42]</td><td colspan="4">Normal
[5.28 6.37 8.28]</td></tr>
<tr><td colspan="4">Example "CNFIncircle(TP1)"</td><td colspan="4">Example "CNFIncircle(TP2)"</td><td colspan="4">Example "CNFIncircle(TP3)"</td></tr>
<tr><td colspan="4">Normal
[4.54 4.54 7.47 8.53]</td><td colspan="4">Normal
[5.01 5.90 5.90 7.33]</td><td colspan="4">Normal
[4.37 4.37 6.66 8.57]</td></tr>
<tr><td colspan="3">Example "CNFIncircle(Diff1)"</td><td colspan="3">Example "CNFIncircle(Diff2)"</td><td colspan="3">Example "CNFIncircle(Diff3)"</td><td colspan="3">Example "CNFIncircle(Diff4)"</td></tr>
<tr><td colspan="3">Normal
[5.07 5.77 5.77 6.93]</td><td colspan="3">Normal
[2.07 3.0 3.00 3.93]</td><td colspan="3">Normal
[5.74 5.74 8.26 8.26]</td><td colspan="3">Normal
[5.13 5.13 5.13 5.13]</td></tr>
</table>

6.4.4. Testing FRI methods based on the PWL benchmark examples

In the following, the FRI methods (KHstabilized-FRI [5], VKK-FRI [4], FRIPOC-FRI [10], and VEIN-FRI [63]) will be compared according to the *PWL_Benchmark Examples* of the KH method. To offer a simple way of comparison, we focused on the cases where the KH-FRI method demonstrated the fails of preserving PWL, which was represented by Examples ("*PWL.NONLIN.C1*" situations1, 2, 3, "*PWL.NONLIN.C2*", "*PWL.NONLIN.C3*" and "*PWL.NONLIN.C4*") as shown in **Table 25 - Table 28.** Therefore, this comparison shows the difference between the selected methods related to the PWL property for each example. Multiple α levels were computed to perform the comparisons. **Fig. 90** introduces the antecedents and observation part of Examples. **Fig. 91- Fig. 94** describe the conclusions of the FRI methods (KHstabilized-FRI [5], VKK-FRI [4], FRIPOC-FRI [10], and VEIN-FRI [63]) and illustrate the difference between the real conclusion (red line) and linearity approximated conclusion (black line).

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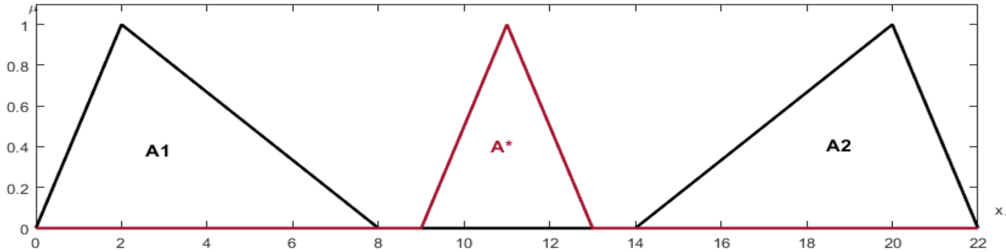
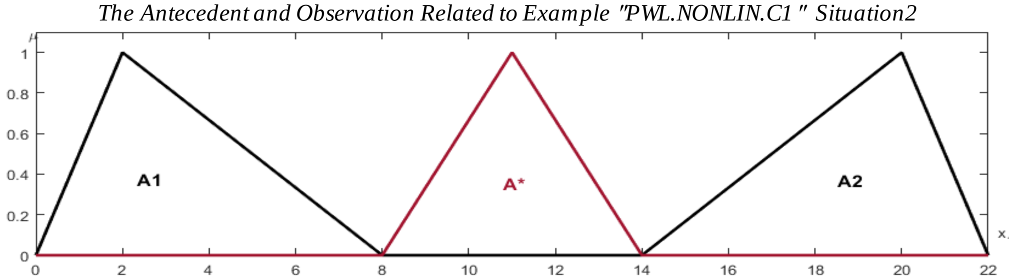
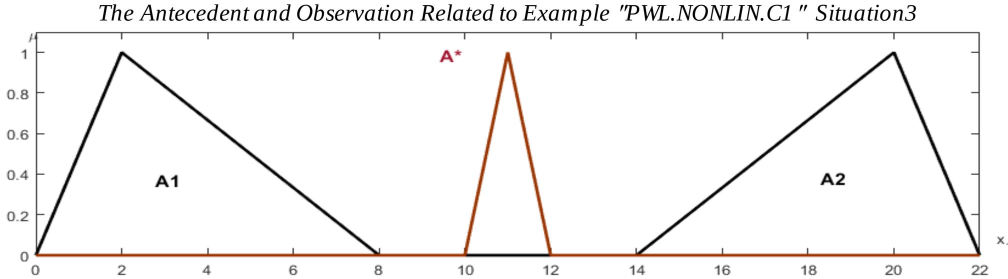
The Antecedent and Observation Related to Example "PWL.NONLIN.C1 " Situation1



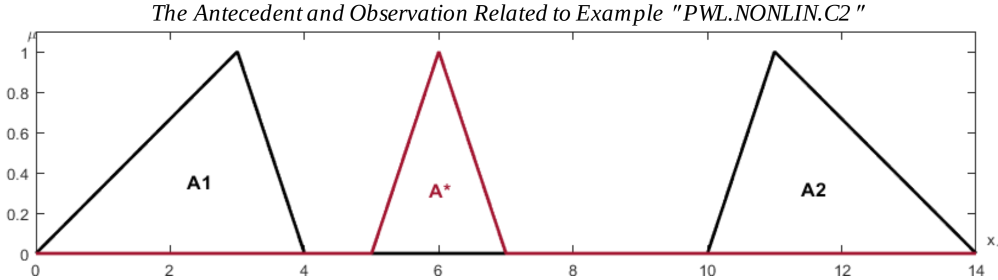
The Antecedent and Observation Related to Example "PWL.NONLIN.C1 " Situation2



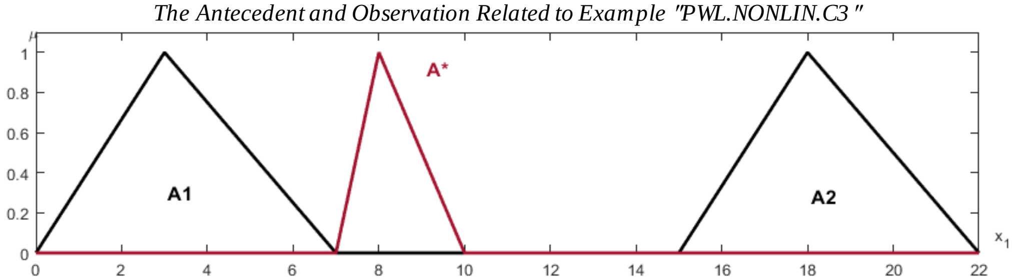
The Antecedent and Observation Related to Example "PWL.NONLIN.C1 " Situation3



The Antecedent and Observation Related to Example "PWL.NONLIN.C2 "



The Antecedent and Observation Related to Example "PWL.NONLIN.C3 "



The Antecedent and Observation Related to Example "PWL.NONLIN.C4 "

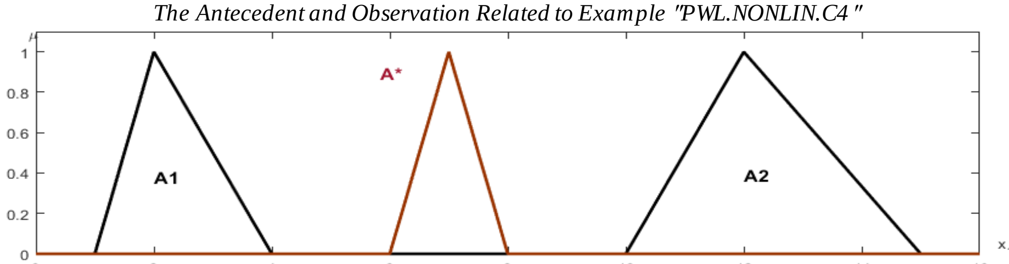


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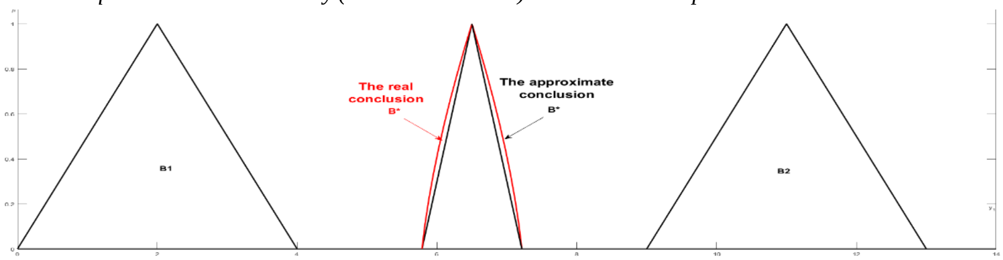
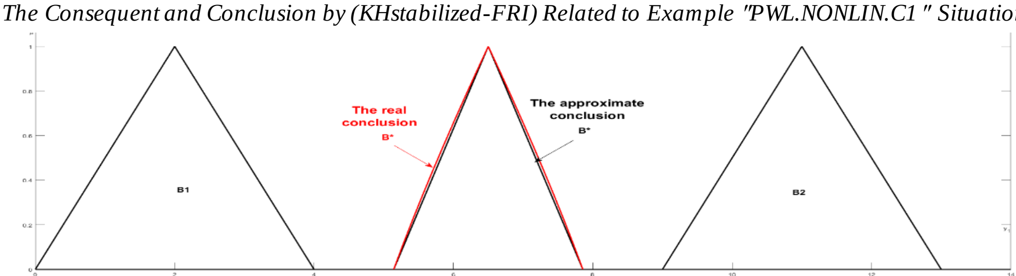
Fig. 90. *The Antecedents and Observations Related to Examples ("PWL.NONLIN.C1 "Situations1,2,3, "PWL.NONLIN.C2 ", "PWL.NONLIN.C3 "and "PWL.NONLIN.C4 ")*

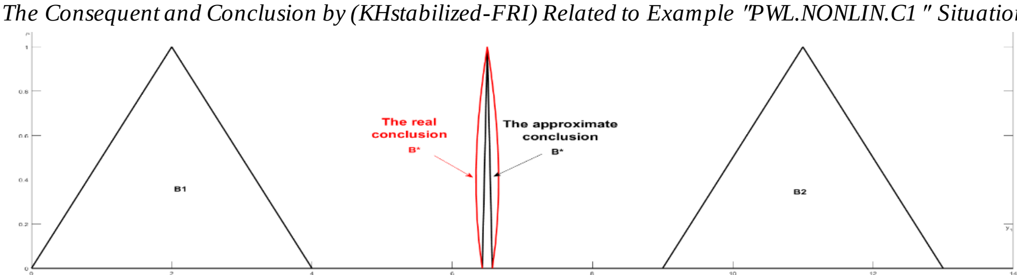
The Consequent and Conclusion by (KHstabilized-FRI) Related to Example "PWL.NONLIN.C1 " Situation1



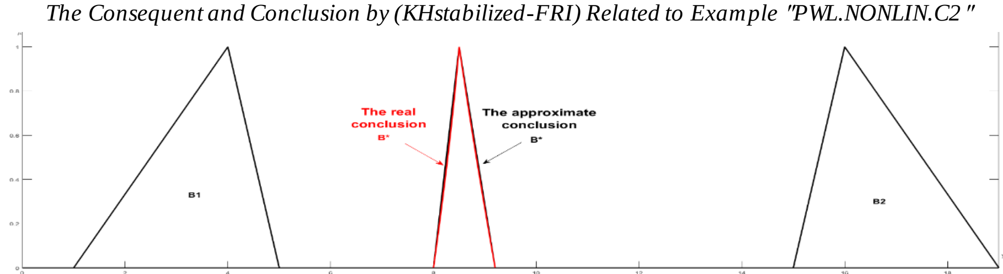
The Consequent and Conclusion by (KHstabilized-FRI) Related to Example "PWL.NONLIN.C1 " Situation2



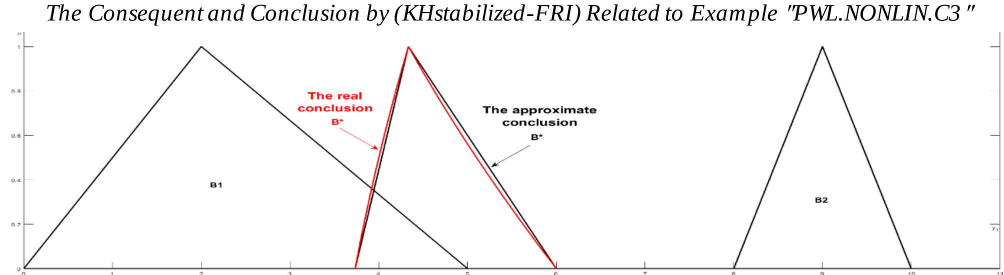
The Consequent and Conclusion by (KHstabilized-FRI) Related to Example "PWL.NONLIN.C1 " Situation3



The Consequent and Conclusion by (KHstabilized-FRI) Related to Example "PWL.NONLIN.C2 "



The Consequent and Conclusion by (KHstabilized-FRI) Related to Example "PWL.NONLIN.C3 "



The Consequent and Conclusion by (KHstabilized-FRI) Related to Example "PWL.NONLIN.C4 "

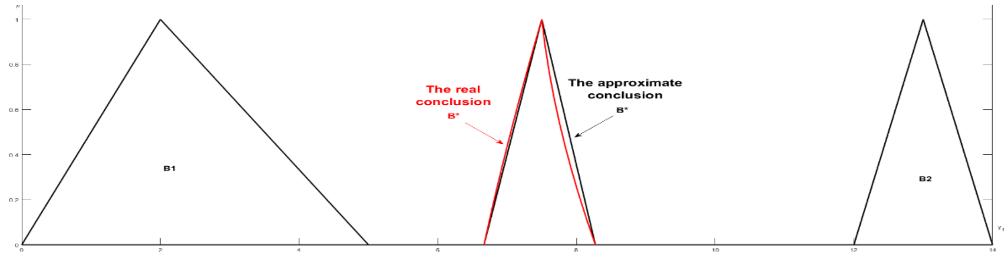
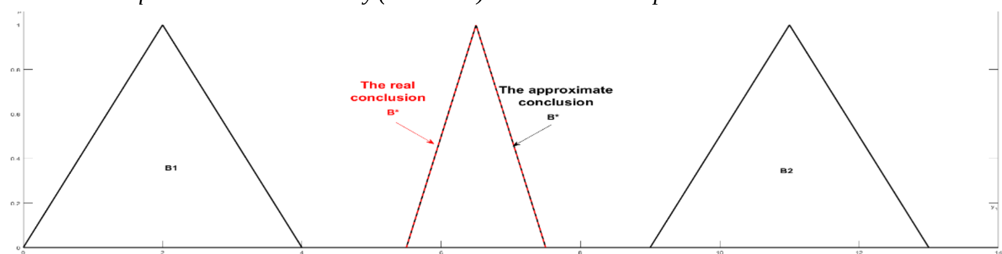
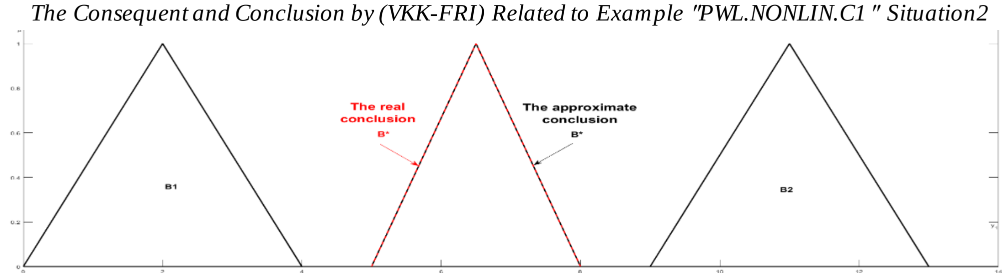


Fig. 91. *The Approximated and Real Conclusions of the KHstabilized-FRI Method to Examples ("PWL.NONLIN.C1 "Situations1,2,3, "PWL.NONLIN.C2 ", "PWL.NONLIN.C3 " and "PWL.NONLIN.C4 ")*

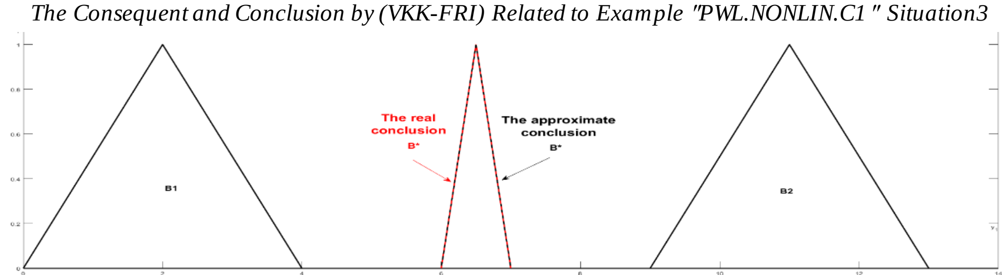
The Consequent and Conclusion by (VKK-FRI) Related to Example "PWL.NONLIN.C1 " Situation1



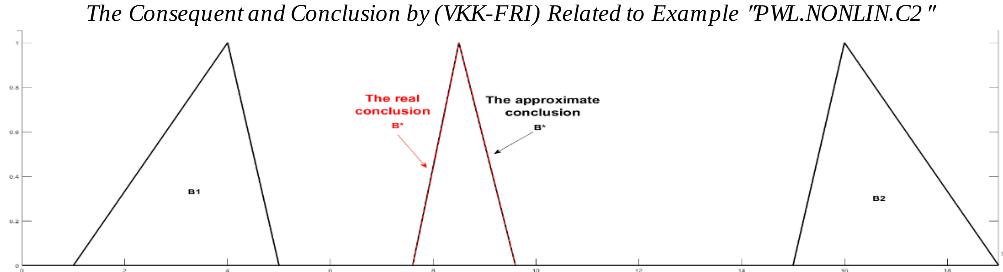
The Consequent and Conclusion by (VKK-FRI) Related to Example "PWL.NONLIN.C1 " Situation2



The Consequent and Conclusion by (VKK-FRI) Related to Example "PWL.NONLIN.C1 " Situation3



The Consequent and Conclusion by (VKK-FRI) Related to Example "PWL.NONLIN.C2 "



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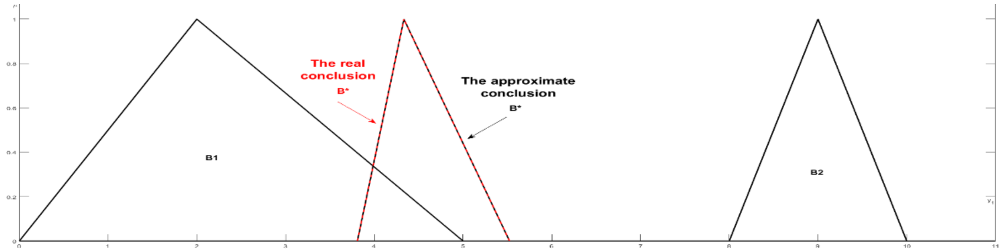
The Consequent and Conclusion by (VKK-FRI) Related to Example "PWL.NONLIN.C3"



The Consequent and Conclusion by (VKK-FRI) Related to Example "PWL.NONLIN.C4"

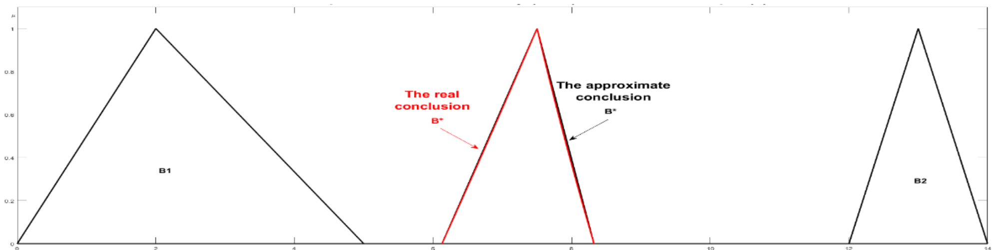
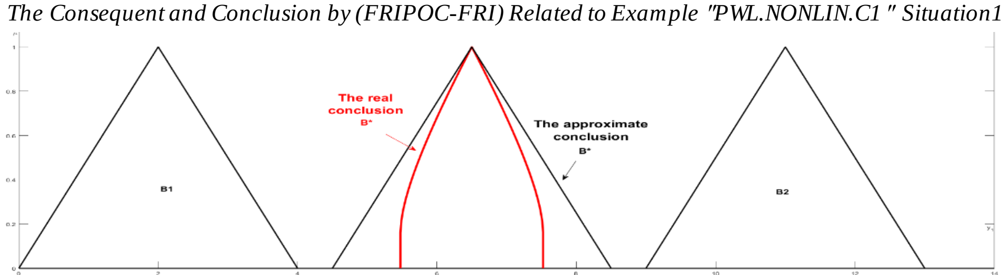
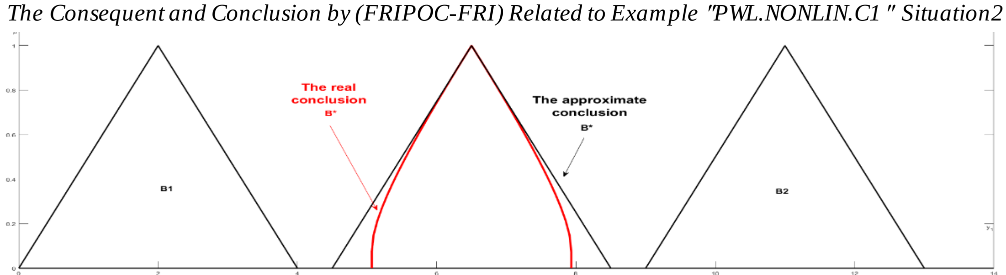


Fig. 92. *The Approximated and Real Conclusions of the VKK-FRI Method to Examples ("PWL.NONLIN.C1" Situations1,2,3, "PWL.NONLIN.C2", "PWL.NONLIN.C3" and "PWL.NONLIN.C4")*

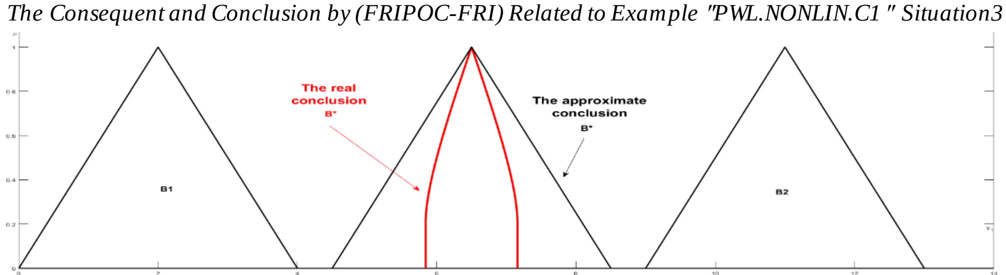
The Consequent and Conclusion by (FRIPOC-FRI) Related to Example "PWL.NONLIN.C1" Situation1



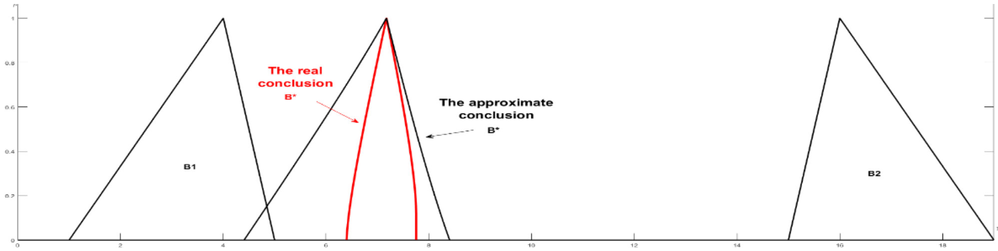
The Consequent and Conclusion by (FRIPOC-FRI) Related to Example "PWL.NONLIN.C1" Situation2



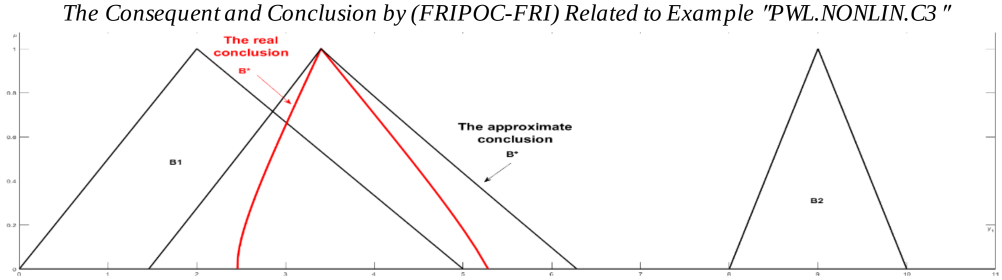
The Consequent and Conclusion by (FRIPOC-FRI) Related to Example "PWL.NONLIN.C1" Situation3



The Consequent and Conclusion by (FRIPOC-FRI) Related to Example "PWL.NONLIN.C2"



The Consequent and Conclusion by (FRIPOC-FRI) Related to Example "PWL.NONLIN.C3"



The Consequent and Conclusion by (FRIPOC-FRI) Related to Example "PWL.NONLIN.C4"

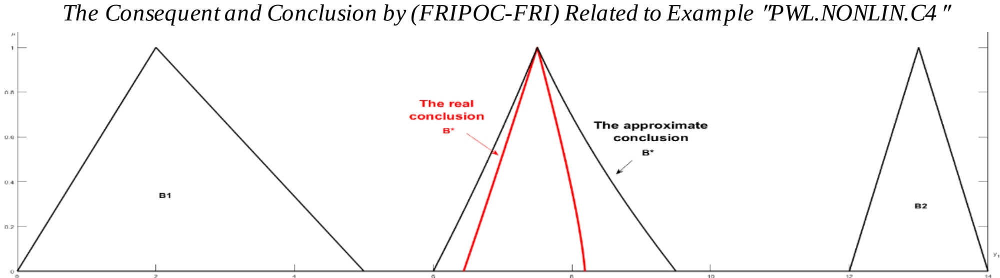
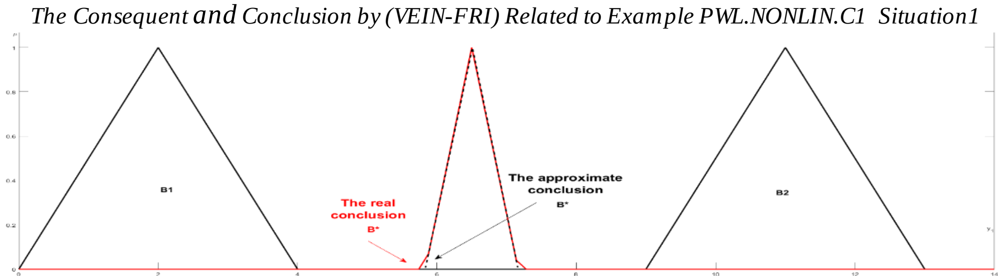
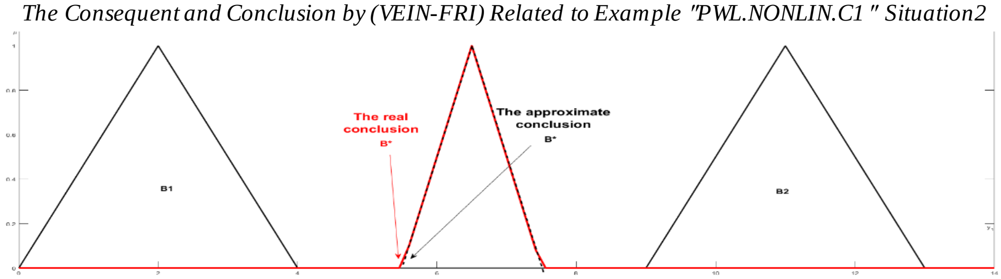


Fig. 93. *The Approximated and Real Conclusions of the FRIPOC-FRI Method to Examples ("PWL.NONLIN.C1" Situations1,2,3, "PWL.NONLIN.C2", "PWL.NONLIN.C3" and "PWL.NONLIN.C4")*

The Consequent and Conclusion by (VEIN-FRI) Related to Example PWL.NONLIN.C1 Situation1



The Consequent and Conclusion by (VEIN-FRI) Related to Example "PWL.NONLIN.C1" Situation2



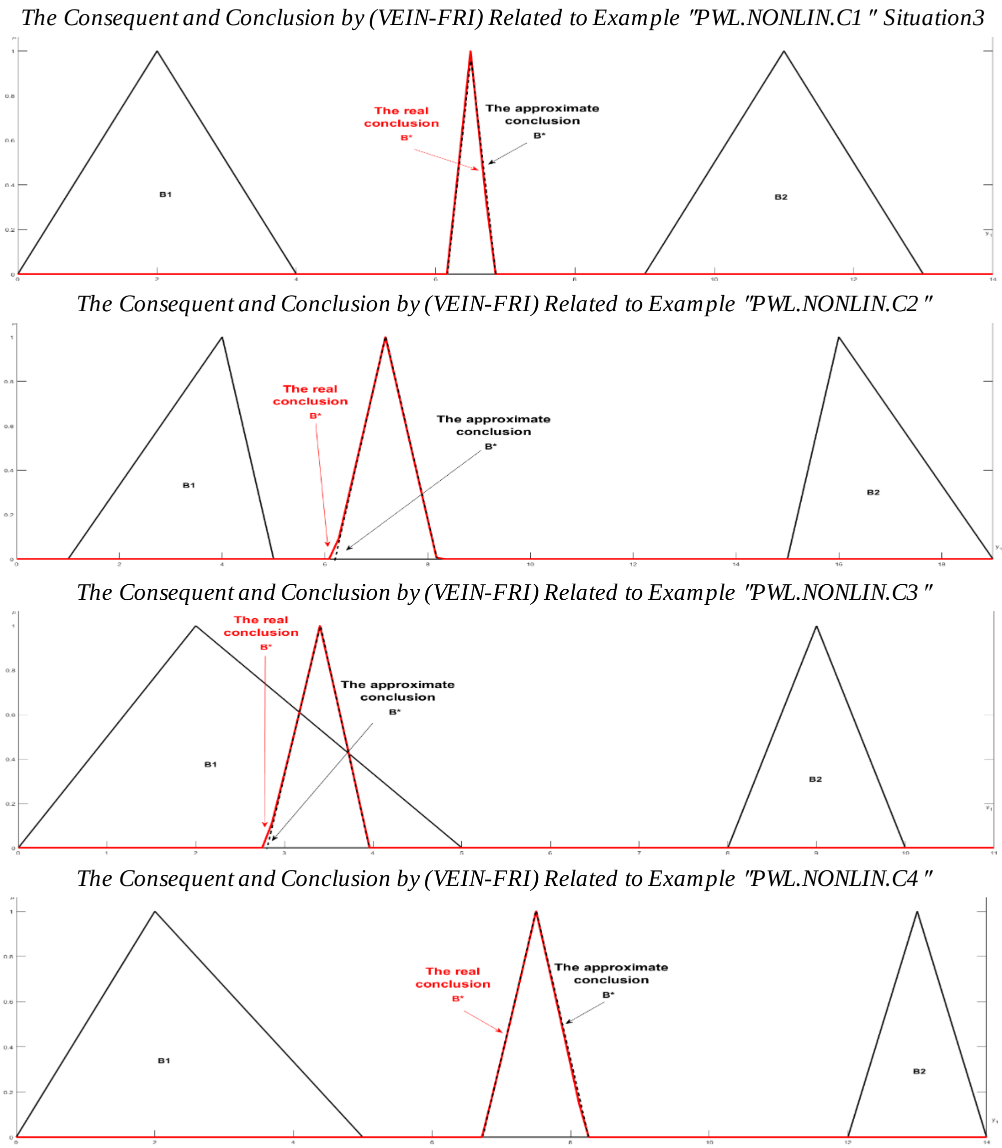


Fig. 94. *The Approximated and Real Conclusions of the VEIN-FRI Method to Examples ("PWL.NONLIN.C1 " Situations1,2,3, "PWL.NONLIN.C2 ", "PWL.NONLIN.C3 "and "PWL.NONLIN.C4 ")*

6.4.5. Testing the Incircle-FRI method based on the PWL benchmark examples

The Incircle-FRI was constructed to preserve the PWL property and handle two α levels in triangular and trapezoidal fuzzy sets. Therefore, according to the *PWL_Benchmark Examples* presented in subsection 6.3, which was constructed to test the PWL property, we examine the Incircle-FRI based on this *PWL_Benchmark* using multi-levels of α-cuts. **Fig. 95** introduces the difference between the real conclusion of the Incircle and linear approximation functions that prove the preservation of the PWL property of the Incircle-FRI.

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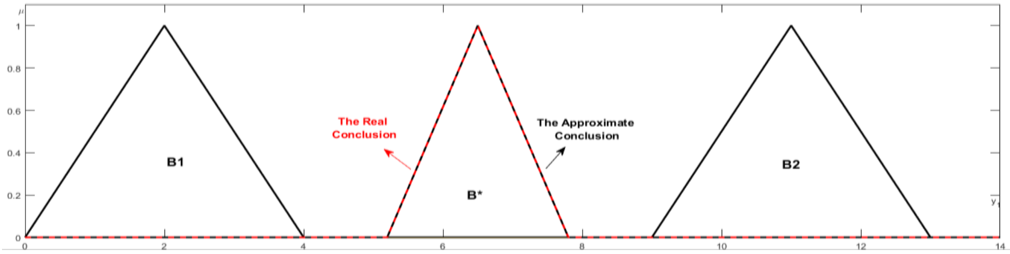
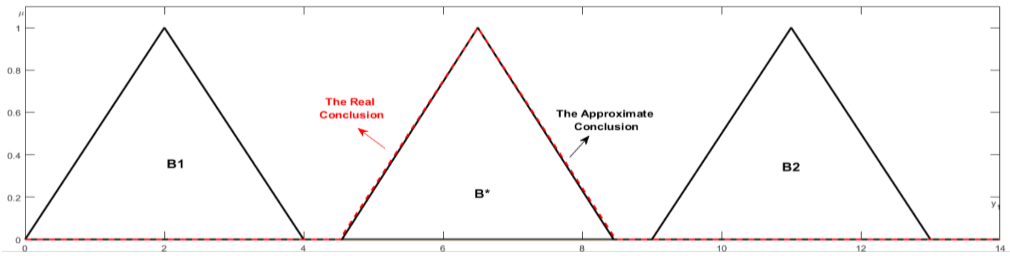
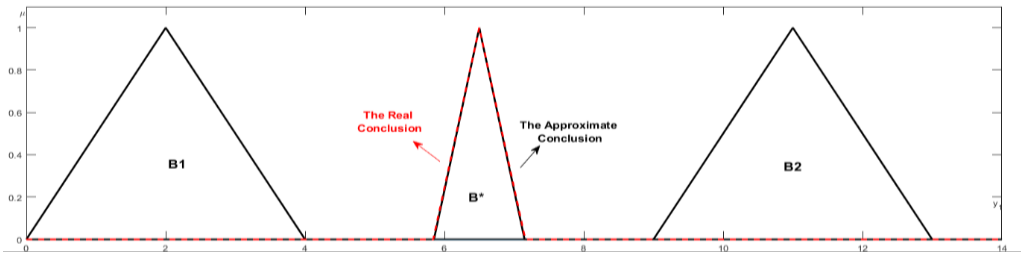
The Consequent and Conclusion by (Incircle-FRI) Related to Example "PWL.NONLIN.C1 " Situation1



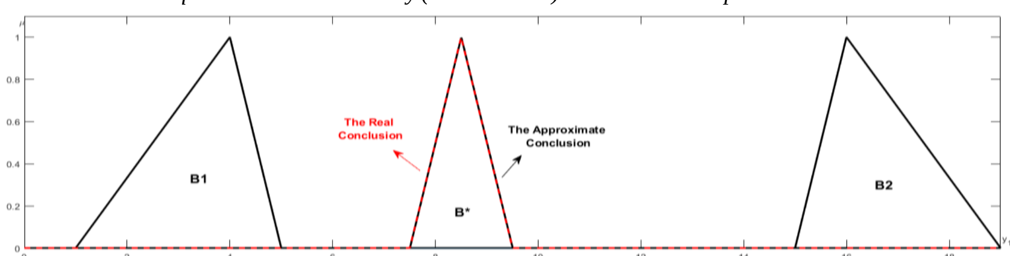
The Consequent and Conclusion by (Incircle-FRI) Related to Example "PWL.NONLIN.C1 " Situation2



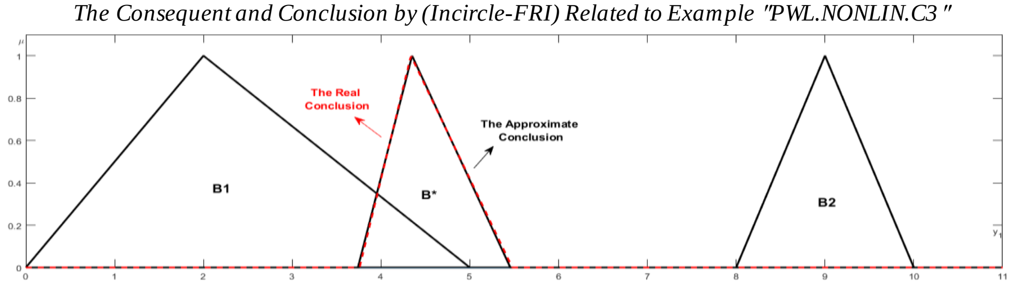
The Consequent and Conclusion by (Incircle-FRI) Related to Example "PWL.NONLIN.C1 " Situation3



The Consequent and Conclusion by (Incircle-FRI) Related to Example "PWL.NONLIN.C2 "



The Consequent and Conclusion by (Incircle-FRI) Related to Example "PWL.NONLIN.C3 "



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The Consequent and Conclusion by (Incircle-FRI) Related to Example "PWL.NONLIN.C4"

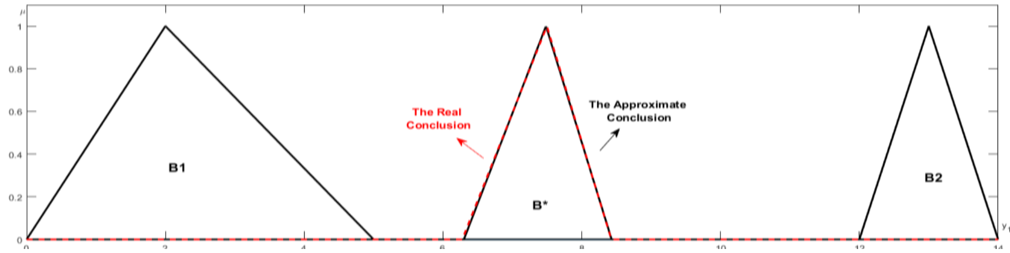

Fig. 95. *The Approximated and Real Conclusions of the Incircle-FRI Method to Examples ("PWL.NONLIN.C1" situations1,2,3, "PWL.NONLIN.C2", "PWL.NONLIN.C3" and "PWL.NONLIN.C4")*

6.4.6. The PWL benchmark examples discussion

According to the results of the FRI methods (KHstabilized-FRI [5], VKK-FRI [4], FRIPOC-FRI [10], VEIN-FRI [63], and Incircle-FRI) (**Fig. 91- Fig. 95**) based on *PWL_Benchmark Examples* ("*PWL.NONLIN.C1*" situations1,2,3, "*PWL.NONLIN.C2*", "*PWL.NONLIN.C3*" and "*PWL.NONLIN.C4*"), we conclude the following:

- KHstabilized-FRI and FRIPOC-FRI methods are not preserving the PWL property to *PWL_Benchmark Examples* ("*PWL.NONLIN.C1*" situations1,2,3, "*PWL.NONLIN.C2*", "*PWL.NONLIN.C3*" and "*PWL.NONLIN.C4*").
- VKK-FRI method succeeded with preserving PWL property in *PWL_Benchmark Examples*, except Example "*PWL.NONLIN.C4*", which appeared with a little bit deviation on the right side.
- VEIN-FRI method succeeded with PWL property on *PWL_Benchmark Examples* ("*PWL.NONLIN.C1*" situations1,2, "*PWL.NONLIN.C2*" and "*PWL.NONLIN.C3*"), in contrast, the Examples ("*PWL.NONLIN.C1*" situation3 and "*PWL.NONLIN.C4*") appeared with a little bit deviation in the bottom boundary.
- Incircle-FRI method is preserving the PWL property on *PWL_Benchmark Examples* ("*PWL.NONLIN.C1*" situations1,2,3, "*PWL.NONLIN.C2*", "*PWL.NONLIN.C3*" and "*PWL.NONLIN.C4*").

Table 46 presents a summary of the results for the selected FRI methods according to the *PWL_Benchmark Examples* ("*PWL.NONLIN.C1*" situations1,2,3, "*PWL.NONLIN.C2*", "*PWL.NONLIN.C3*" and "*PWL.NONLIN.C4*") to PWL property, where the plus sign (+) indicates that the technique is satisfied with PWL property, while a minus sign (-) shows the method has a little bit deviation in some cases. The cross sign (x) indicates that the method did not preserve the PWL property.

Table 46. *Summary of the FRI Methods and Their Conformity to PWL_Benchmark Examples ("PWL.NONLIN.C1 "situations1,2,3, "PWL.NONLIN.C2", "PWL.NONLIN.C3 "and "PWL.NONLIN.C4").*

Examples	Methods					
	KH FRI [3],[25],[39]	KHStabilized FRI [5]	VKK FRI [4]	FRIPOC FRI [10]	VEIN FRI [63]	Incircle FRI
"PWL.NONLIN.C1" situation1	x	x	+	x	-	+
"PWL.NONLIN.C1" situation2	x	x	+	x	-	+
"PWL.NONLIN.C1" situation3	x	x	+	x	+	+
"PWL.NONLIN.C2"	x	x	+	x	-	+
"PWL.NONLIN.C3"	x	x	+	x	-	+
"PWL.NONLIN.C4"	x	x	-	x	+	+

SUMMARY

Regarding the general conditions and criteria that have been suggested for unifying the common requirements, FRI methods have to be satisfied. One of the most common conditions is the demand for a Convex and Normal Fuzzy (CNF) conclusion in case all the rule antecedents and consequents are CNF sets, and another condition is the fuzzy set of the conclusion must preserve a Piecewise Linearity (PWL), in case all antecedents and consequents of the fuzzy rules are preserving PWL sets at α-cut levels. The KH-FRI is the one, which cannot fulfill these conditions. The main goals of this chapter were:

Introduced different arbitrary examples to compare FRI techniques based on the various features: No. of Dimensions, Type of Membership Functions, and No. of Membership Functions for the antecedent and consequent, as presented in **Table 13**. These arbitrary examples used to classify and compare FRI methods based on the criteria of the normality and linearity properties. The results of the arbitrary examples were described as follows: KH, KH Stabilized, LESFRI, and VKK methods suffered from the abnormality, in case of having multi-dimension antecedents and different type of membership functions which was described in Examples (FRI_EX6 and FRI_EX7). The VKK method suffered from the abnormality, in case of having single-dimension, as shown in Example (FRI_EX3). The FRIPOC method suffered from non-preserve piecewise linearity in case of multi-dimension antecedents and different types of membership functions, as shown in Example (FRI_EX6). In contrast, MACI, IMUL, CRF, GM, and SCALE MOVE methods did not suffer from abnormality and piecewise linearity for all arbitrary examples.

Investigate equations and notations related to CNF and PWL properties, which aim to highlight the problematic properties of the KH-FRI method to prove its efficiency with CNF and PWL properties. This chapter was focusing on constructing benchmark examples to be a baseline for testing other FRI methods against situations that are not satisfied with the normality and linearity properties for KH-FRI. The *CNF_Benchmark Examples* were created based on different examples ("CNF.KH_ABNOR.C4_1" – "CNF.KH_ABNOR.C4_4") and their cases that based on (the core and boundary of the rule-bases and observation fuzzy sets), it proved the conclusion of the KH-FRI is always abnormal. While the *PWL_Benchmark Examples* that were constructed based on different examples ("PWL.NONLIN.C1" situations1,2,3, "PWL.NONLIN.C2", "PWL.NONLIN.C3" and "PWL.NONLIN.C4") and their cases that based on (the left and right slopes of the rule-bases and observation fuzzy sets), which determine the conclusion of the KH-FRI is not preserving on the linearity. Moreover, this chapter proved the efficiency of the benchmark CNF and PWL properties to examine some FRI methods. Concerning the proposed Incircle-FRI method, it proved that it is a suitable approach to be implemented as an inference system because its conclusions succeeded with CNF and PWL properties on all CNF and PWL benchmark examples.

Thesis related to Chapter 6:

Thesis. IV:

I introduced the initial benchmark system (set of benchmark examples) for the most important properties of the FRI concept (CNF and PWL properties), constructing benchmarks are based on analyzing all cases of the core, boundary, and slopes conditions of the antecedents, consequents, and observation fuzzy sets. The KH-FRI CNF and PWL Benchmark are suitable for highlighting some problematic points of the KH-FRI and other FRI methods that originated from the KH-FRI. Therefore, CNF and PWL Benchmarks are suitable for evaluating and comparing FRI methods, where the KH-FRI is not satisfied with CNF and PWL properties.

The results introduced in this chapter are supporting the statement of Thesis IV and [32], [35], [94]

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The dissertation's main contribution is the proposal of a new fuzzy interpolative reasoning method based on the properties of the Incircle triangular fuzzy number. The suggested method is based on the center point of the incircle triangular fuzzy number as a reference point of the fuzzy set. The main sides of the triangular are indicated SD_1, SD_2, and SD_3 (see **Fig. 96** for Incircle_Notations). The tangents length and vertices of the triangle with its Incircle, which denotes PS_1, PS_2, and PS_3, referred to as "*fuzziness sides*". The proposed Incircle-FRI is always producing triangular CNF fuzzy conclusion by holding the same rate of the weights among the observation and the two rule antecedents, and the conclusion and the two corresponding rule consequents with the reference points, and with the "fuzziness sides" (see **Fig. 97**).

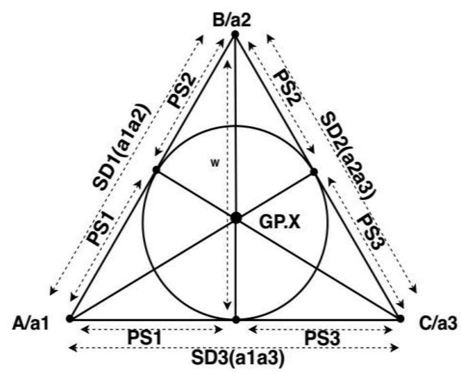

Fig. 96. *Triangular Fuzzy Number Incircle_Notations.*

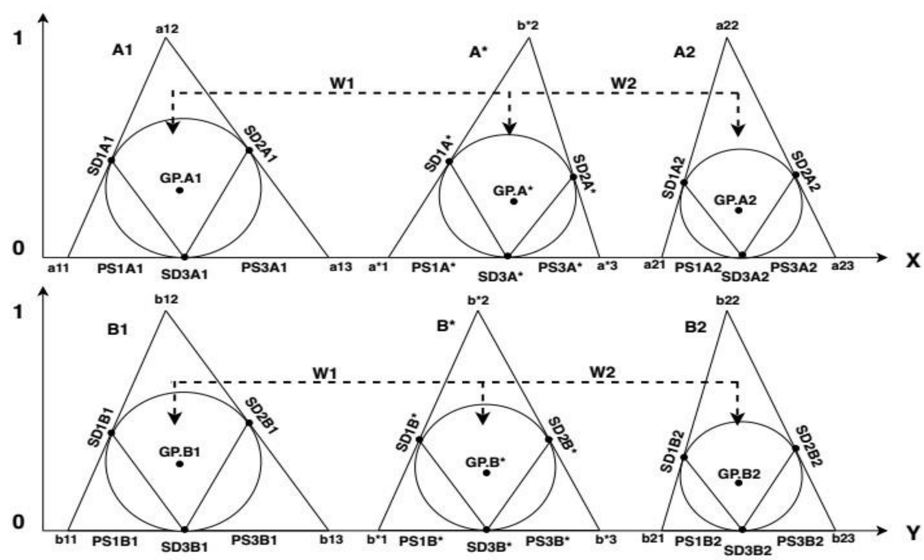

Fig. 97. *Fuzzy Interpolative Reasoning using Triangular Membership Functions*

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The proposed Incircle-FRI can also be extended to handle singleton, trapezoidal (see **Fig. 98**), and hexagonal (see **Fig. 99**) membership functions. The proposed Incircle-FRI method can also extend to be able to handle fuzzy interpolative reasoning with multiple antecedent variables and multiple fuzzy rules. Extending the Incircle-FRI with general weight calculation and a shift process, the suggested FRI method can also perform extrapolation. (see Chapter 4, 5 for the details and the related Thesis I, II, and III.)

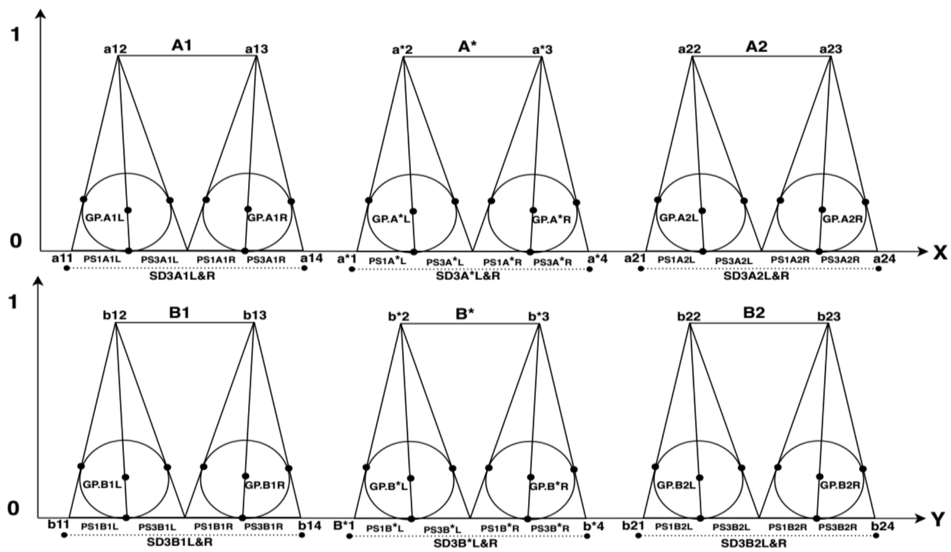

Fig. 98. *Fuzzy Interpolative Reasoning using Trapezoidal Membership Functions*

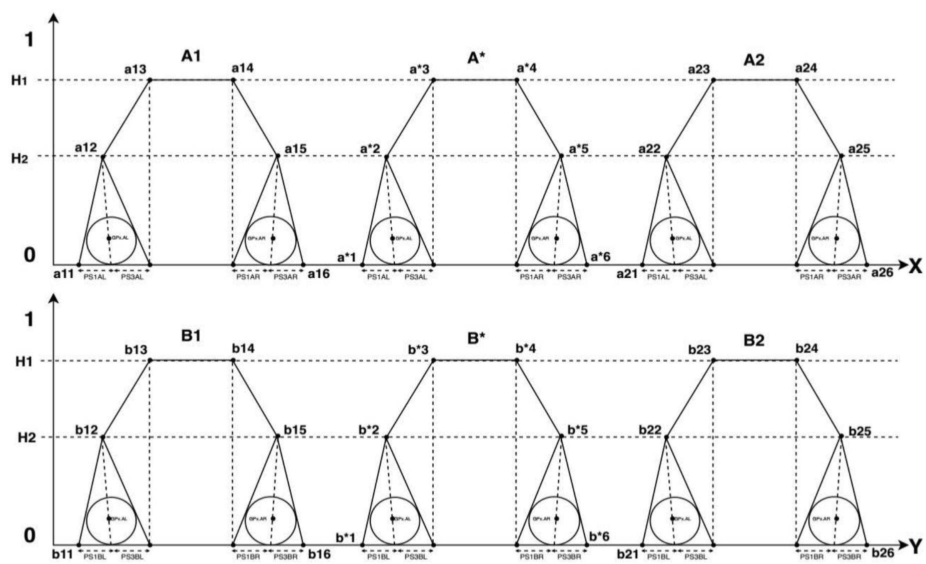

Fig. 99. *Fuzzy Interpolative Reasoning using Hexagonal Membership Functions*

Another important contribution of the dissertation is the proposal of a novel FRI benchmark system. The suggested benchmark is far not ready. The *CNF* (Convex and Normal Fuzzy) and *PWL* (Piecewise Linearity) benchmark examples are just the first step and a methodology to construct a comprehensive FRI (Fuzzy Rule Interpolation) benchmark system, which is built upon the weaknesses of the existing FRI methods, and can also highlight the strengths of the newcomer ones. (See Chapter 4, 6 for the details and the related Thesis IV.)

The main scientific results of the research presented in this work are summarized in the following Theses:

Thesis. I:
I introduced a new method for the fuzzy rule interpolation concept called "Incircle-FRI", which is based on the Incircle of a triangular fuzzy number, the Gergonne Point as a "reference point" of the inside circle of triangular fuzzy set, and the fuzziness sides of the triangular. The Incircle-FRI conclusion is calculated by holding the same rate of the weights among the observation and the two rule antecedents, and the conclusion and the two corresponding rule consequents with the Gergonne Points (for the reference point of the conclusion), and with the "fuzziness sides" (for left and right fuzziness the shape of the conclusion). The "Incircle-FRI" is always generating a triangular CNF conclusion, if the antecedents and the consequents are triangular CNF sets, even if the fuzzy rule-base is sparse. I conclude that the proposed method is a suitable approach to be implemented as an inference system.

Thesis. II:
I introduced an extension of the "Incircle-FRI" to be able to handle trapezoidal and hexagonal fuzzy sets, which is by decomposing their membership function shapes into multiple triangulars, and multiple Incircle triangular fuzzy numbers with the Gergonne Points as reference points. I conclude that the extended "Incircle-FRI" can generate a trapezoidal, or hexagonal CNF conclusion if the antecedents and the consequents are all trapezoidal, or all hexagonal CNF sets, even if the fuzzy rule-base is sparse. Therefore, the Incircle-FRI method is a suitable approach to be implemented as an inference system with trapezoidal and hexagonal fuzzy sets.

Thesis. III:
I introduced an extension of the "Incircle-FRI" to be able to handle multiple fuzzy rules having multiple fuzzy antecedents. I used a modification weight estimate and included a shift technique to ensure to interpolate the consequent fuzzy result to be more logical and also to enable the capability for extrapolation. I conclude that the extensions of the "Incircle-FRI" always produce CNF conclusion, for all the handled antecedents and consequents configuration of the original method even if the fuzzy rule-base is sparse. Therefore, the Incircle-FRI method is a suitable approach to be implemented as an inference system with these extensions.

Thesis. IV:

I introduced the initial benchmark system (set of benchmark examples) for the most important properties of the FRI concept (CNF and PWL properties), constructing benchmarks are based on analyzing all cases of the core, boundary, and slopes conditions of the antecedents, consequents, and observation fuzzy sets. The KH-FRI CNF and PWL Benchmark are suitable for highlighting some problematic points of the KH-FRI and other FRI methods that originated from the KH-FRI. Therefore, CNF and PWL Benchmarks are suitable for evaluating and comparing FRI methods, where the KH-FRI is not satisfied with CNF and PWL properties.

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APPENDIX A

Appendix A.1

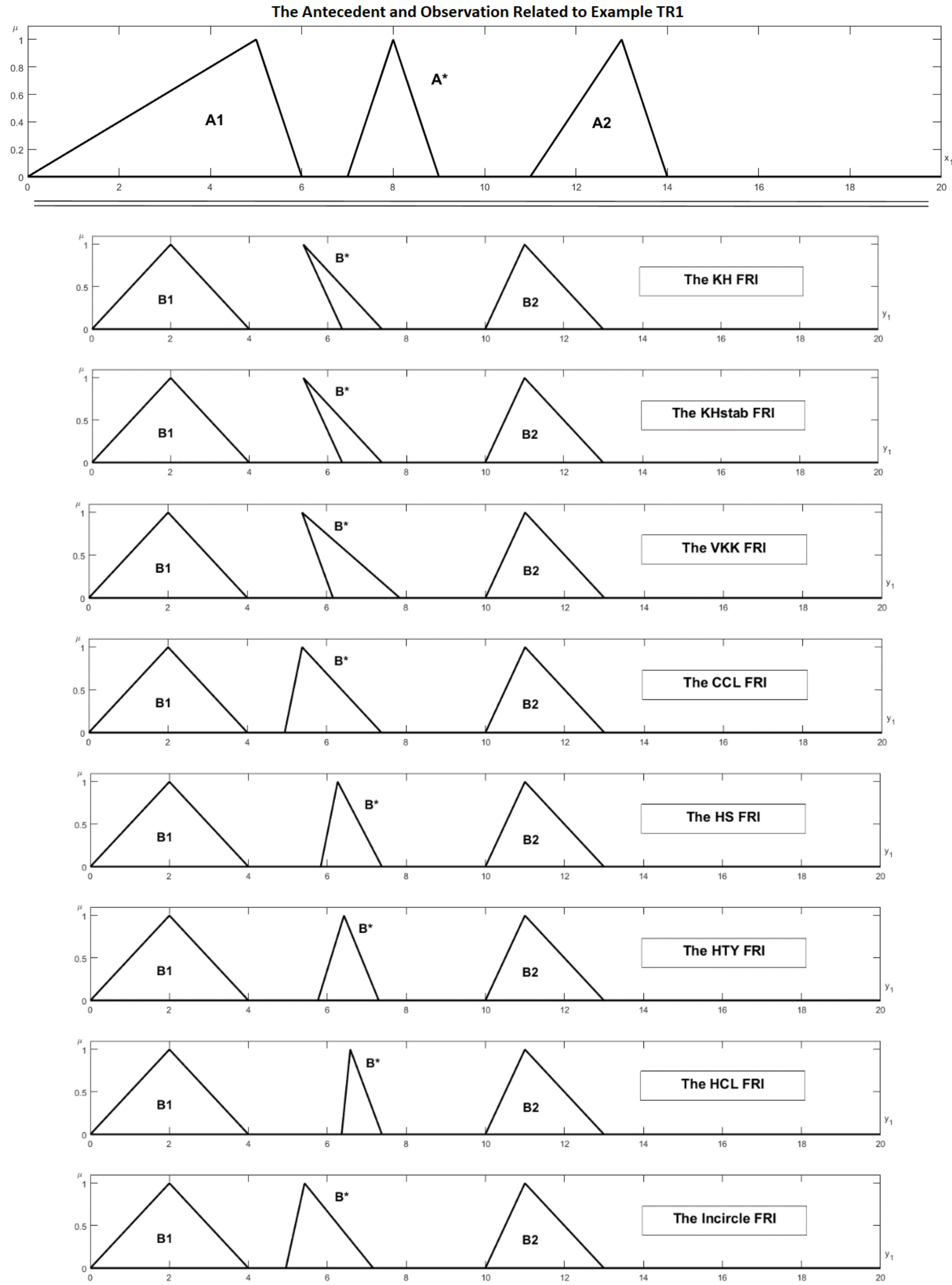

Fig. 100. *A Comparison of Fuzzy Interpolative Reasoning Results of Example TR1 for Several FRI Methods.*

Appendix A.2

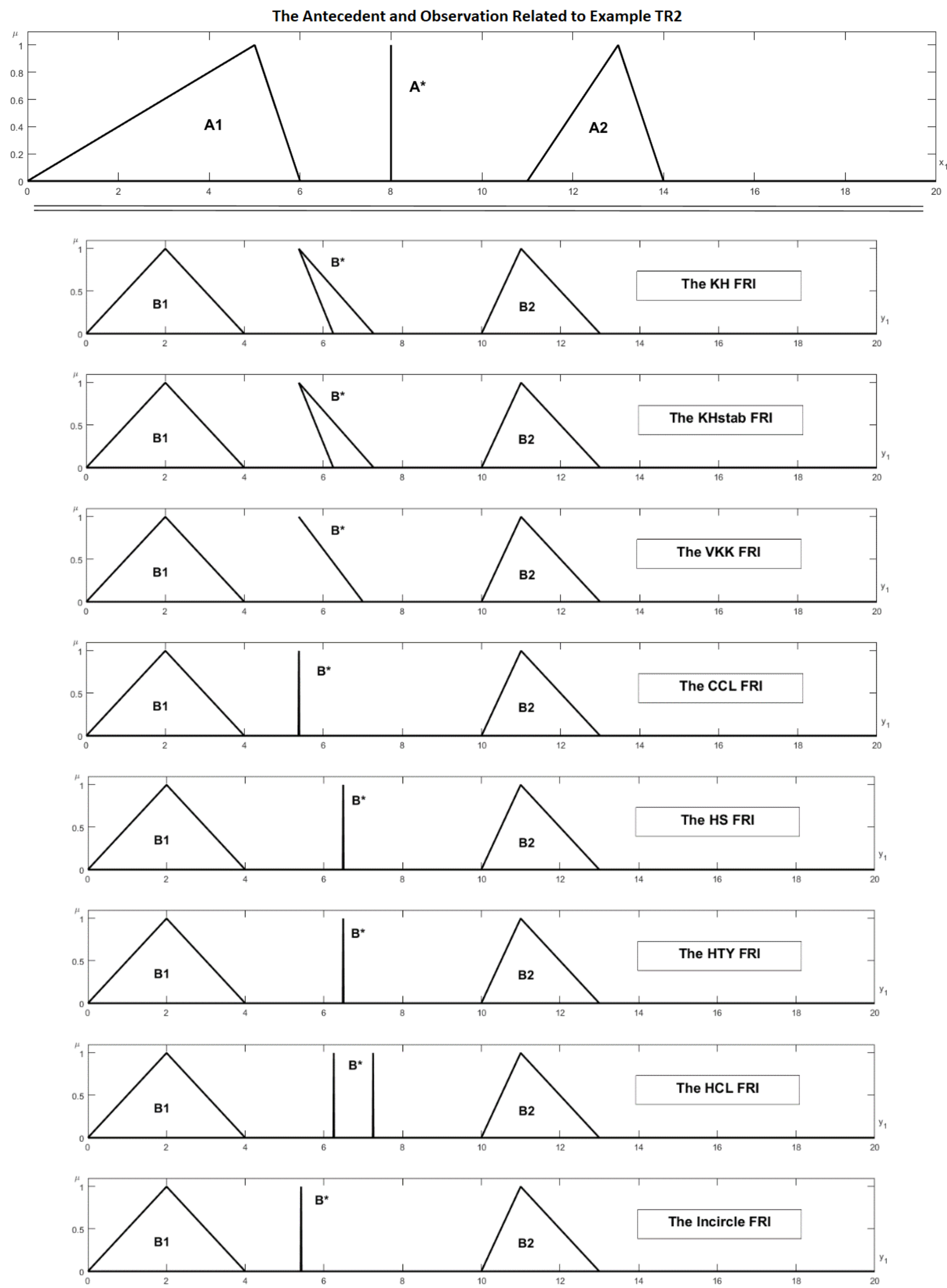

Fig. 101. *A Comparison of Fuzzy Interpolative Reasoning Results of Example TR2 for Several FRI Methods.*

Appendix A.3

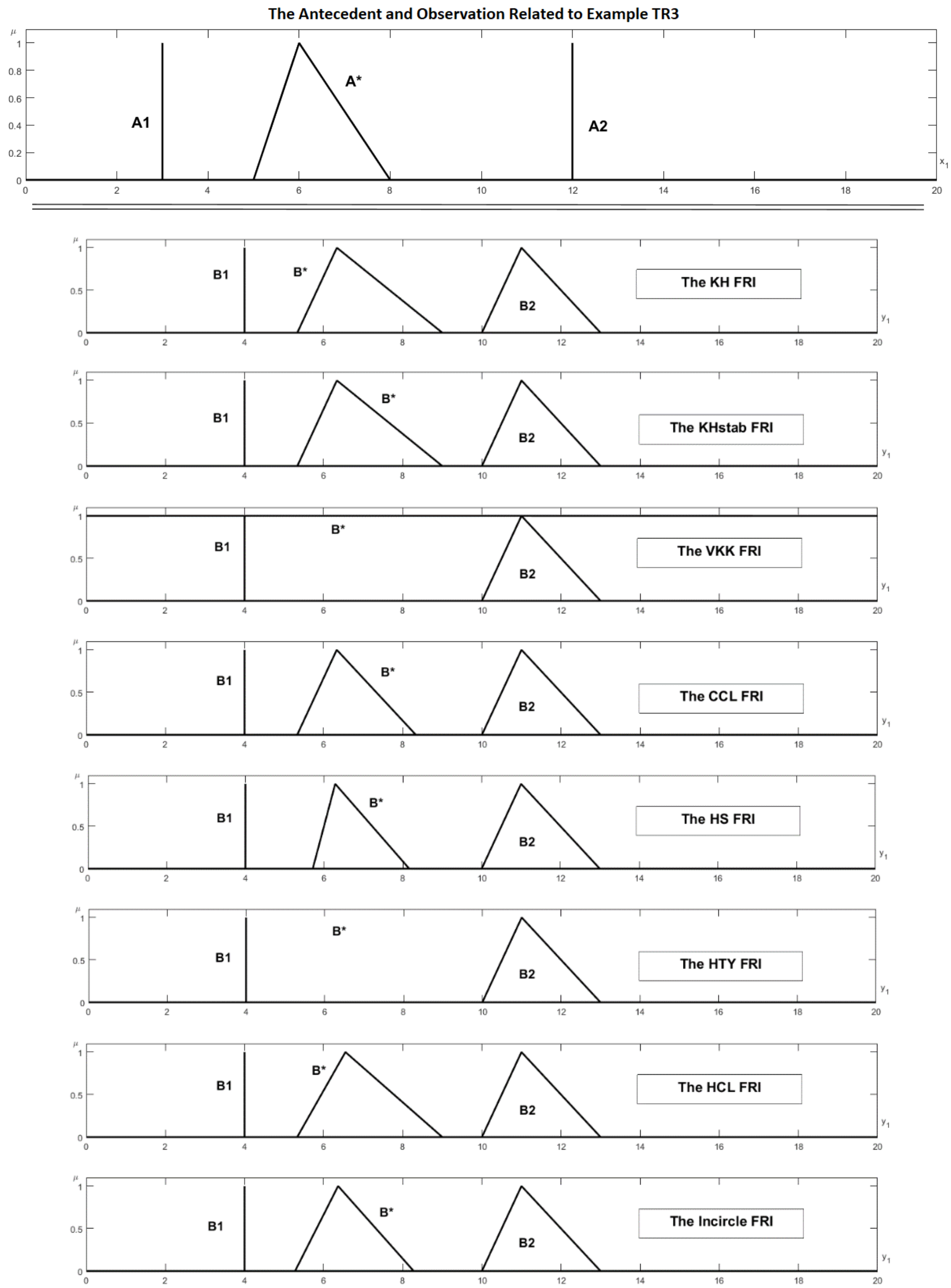

Fig. 102. *A Comparison of Fuzzy Interpolative Reasoning Results of Example TR3 for Several FRI Methods.*

Appendix A.4

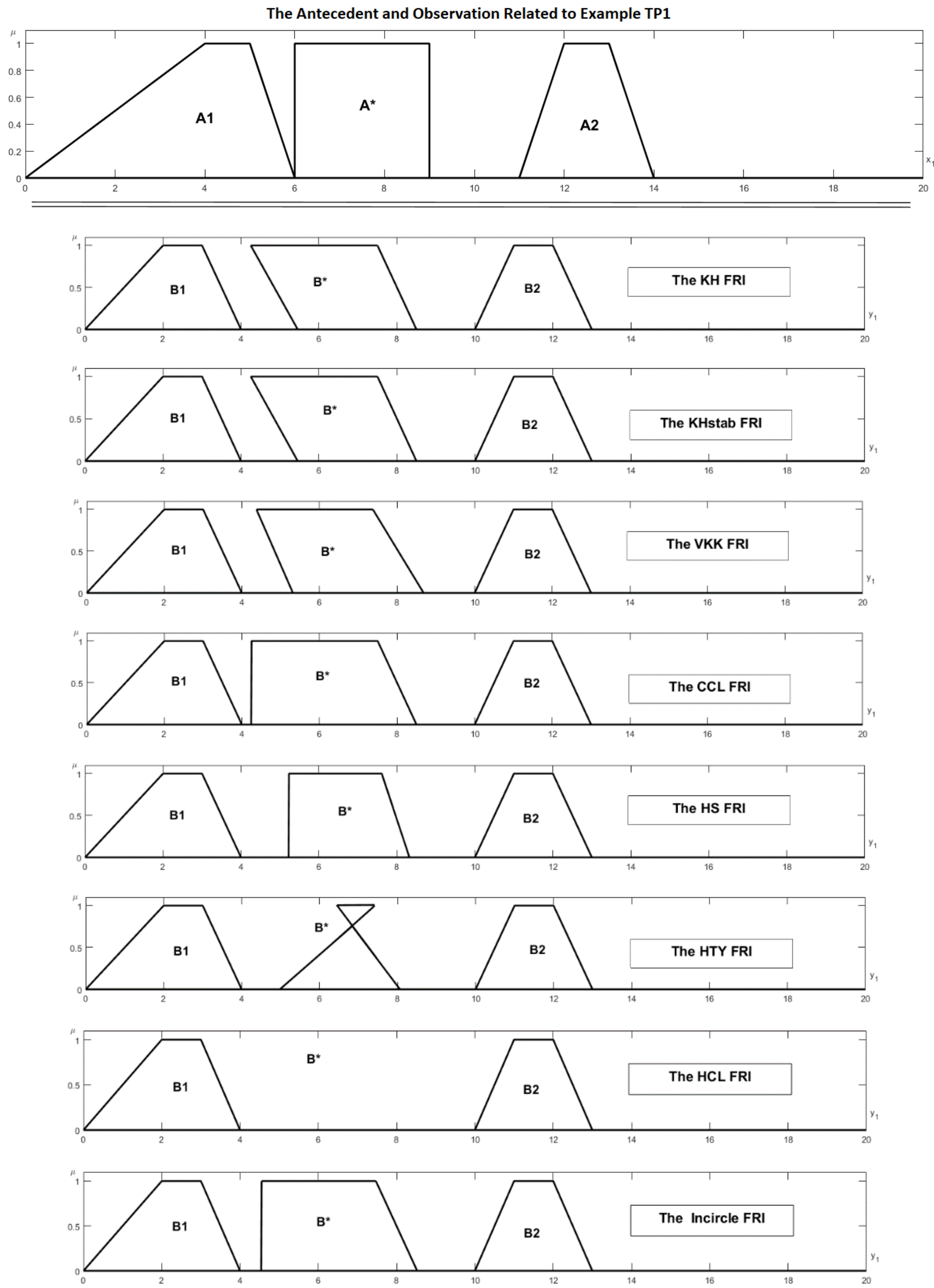

Fig. 103. A Comparison of Fuzzy Interpolative Reasoning Results of Example TP1 for Several FRI Methods.

Appendix A.5

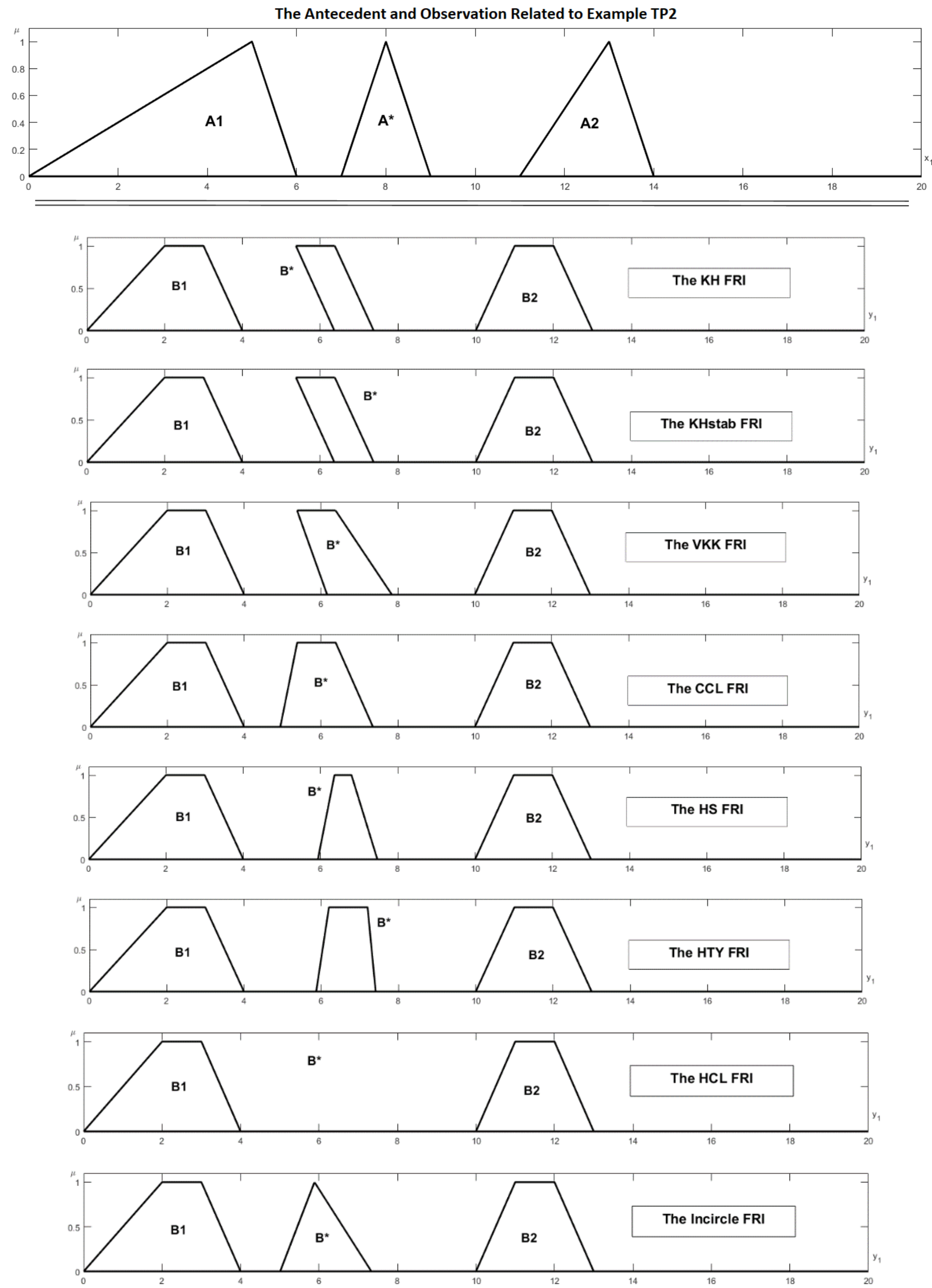

Fig. 104. *A Comparison of Fuzzy Interpolative Reasoning Results of Example TP2 for Several FRI Methods.*

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Appendix A.6

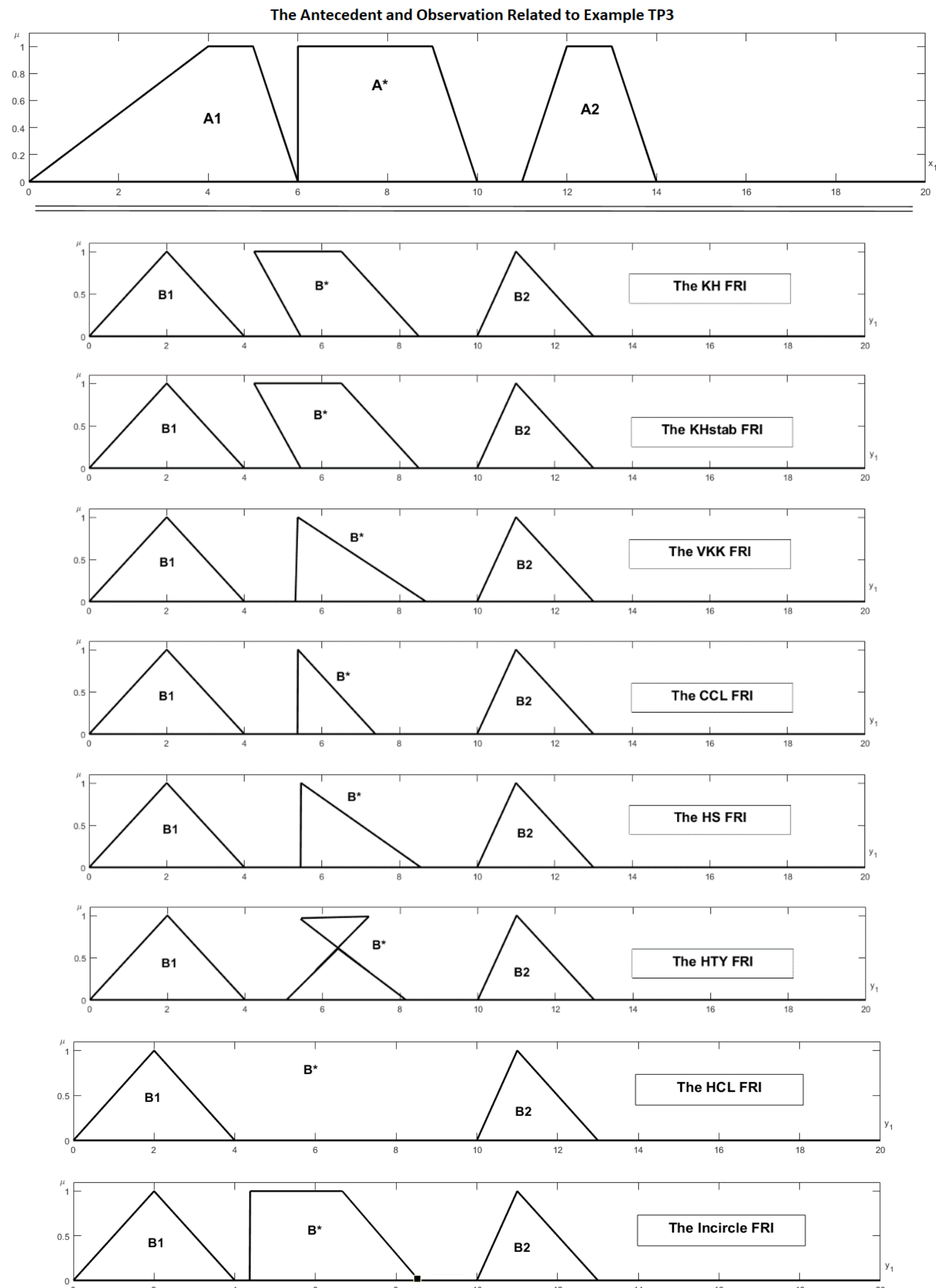

Fig. 105. *A Comparison of Fuzzy Interpolative Reasoning Results of Example TP3 for Several FRI Methods.*

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Appendix A.7

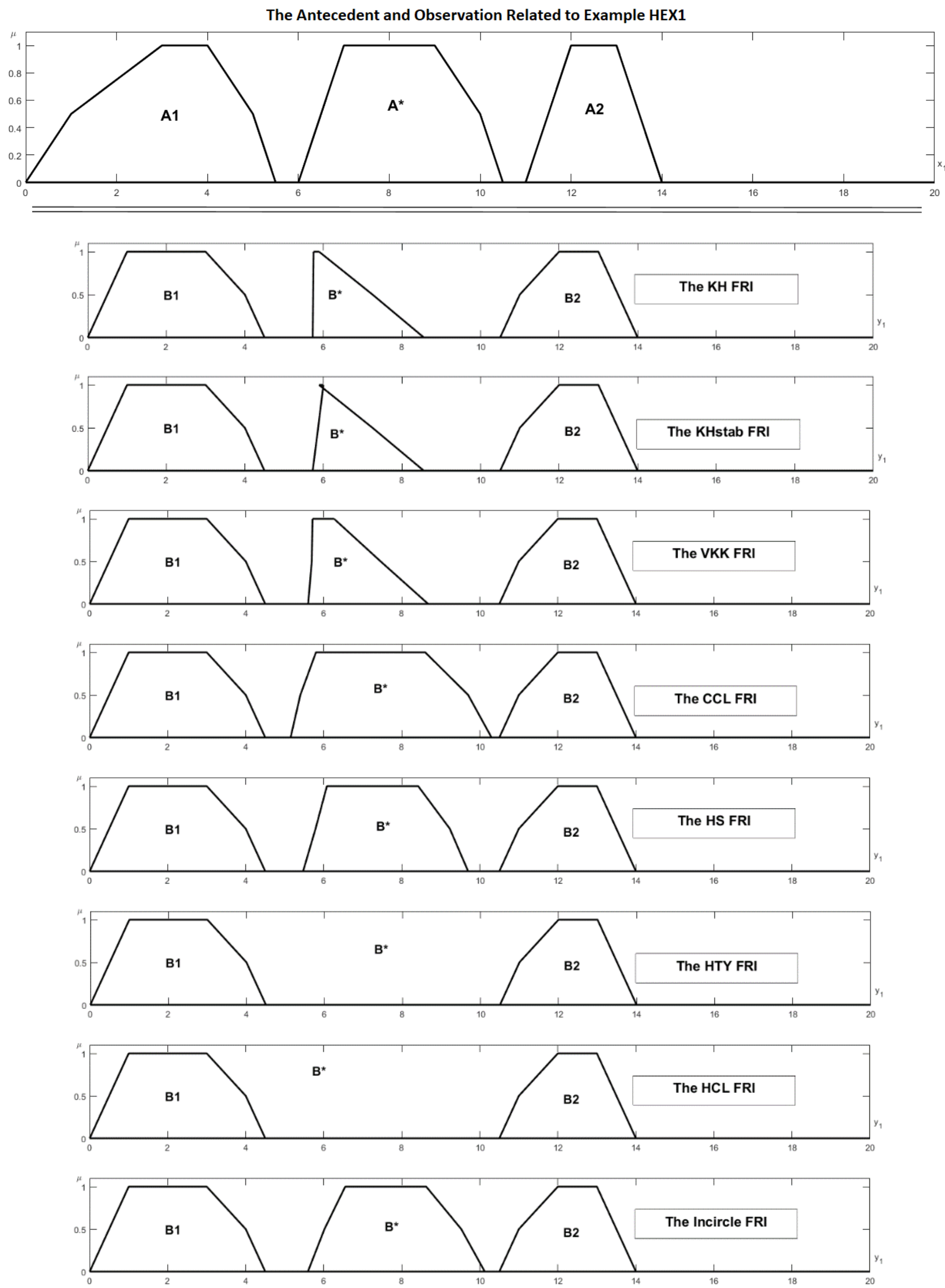

Fig. 106. *A Comparison of Fuzzy Interpolative Reasoning Results of Example HEX1 for Several FRI Methods.*

Appendix A.8

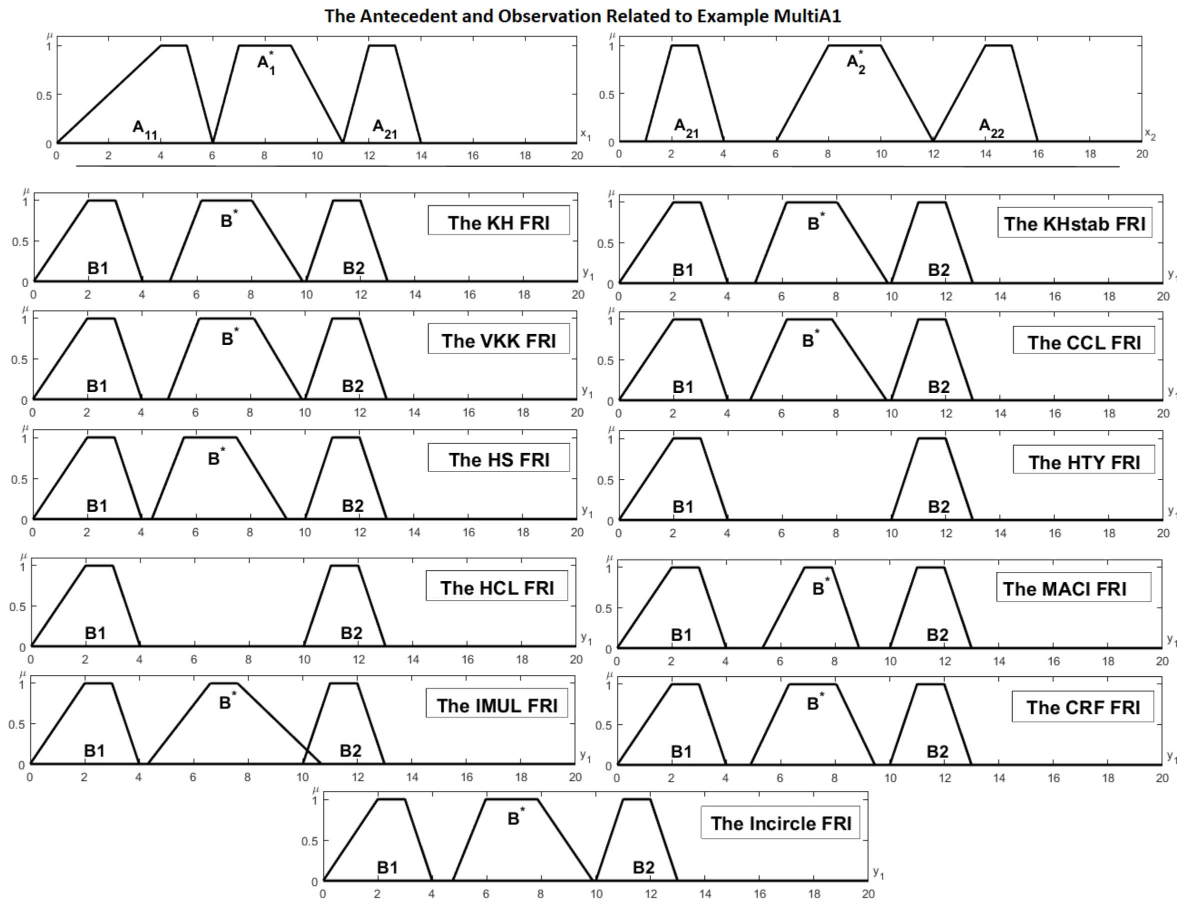

Fig. 107. *A Comparison of Fuzzy Interpolative Reasoning Results of Example MultiA1 for Several FRI Methods.*

Appendix A.9

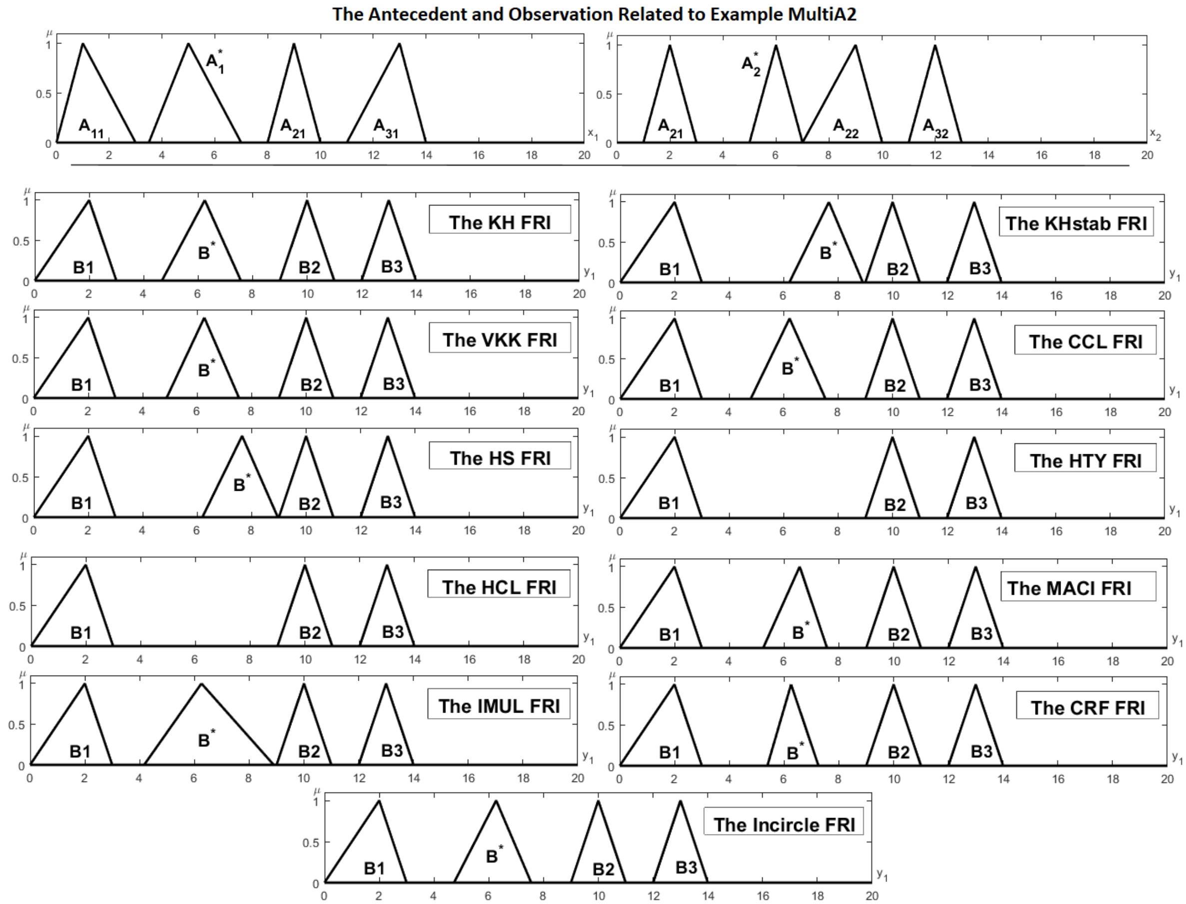

Fig. 108. *A Comparison of Fuzzy Interpolative Reasoning Results of Example MultiA2 for Several FRI Methods.*

Appendix A.10

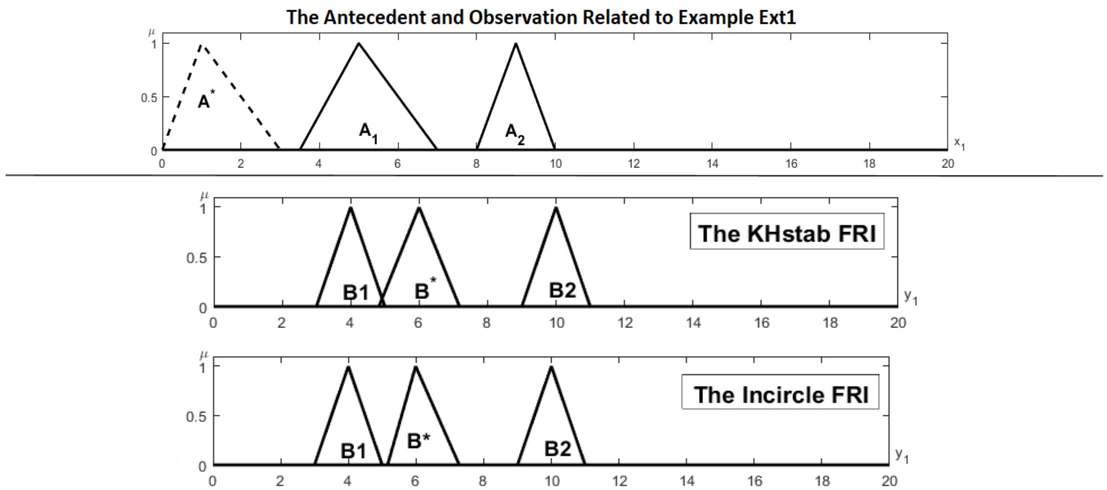

Fig. 109. A Comparison of Fuzzy Interpolative Reasoning Results of Example Ext1 for Several FRI Methods.

Appendix A.11

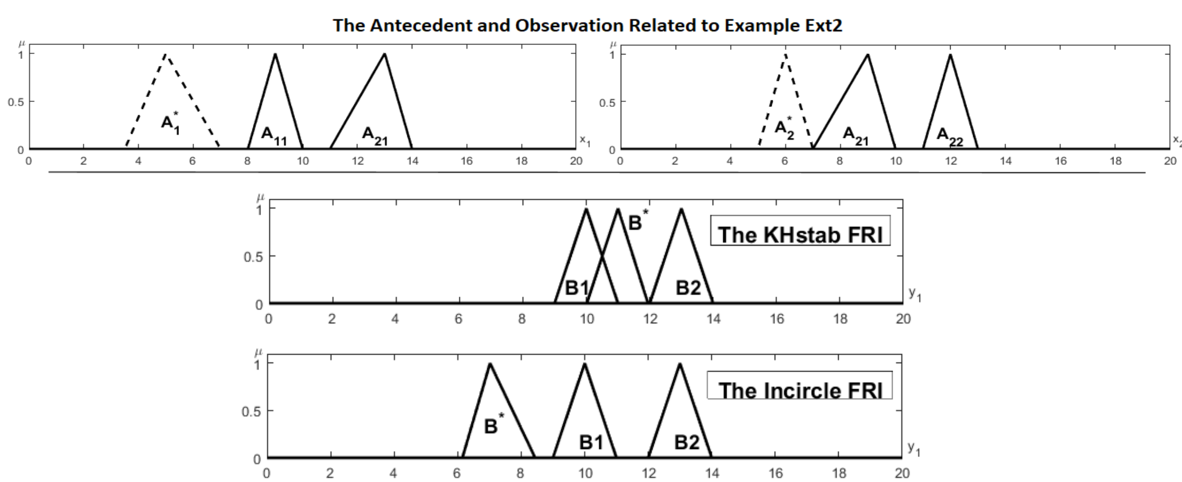

Fig. 110. A Comparison of Fuzzy Interpolative Reasoning Results of Example Ext2 for Several FRI Methods.

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APPENDIX B

A Publications Arising from the Dissertation

A few publications have been generated from the research carried out within the PhD project. Below list the resultant publications that are in close relevance to the dissertation, including all the papers already published.

*[32]. M. Alzubi, Z. C. Johanyak, and S. Kovacs, "**Fuzzy Rule Interpolation Methods and FRI Toolbox**", **Journal Scopus Indexed [Q3].***

*[35]. M. Alzubi and S. Kovacs, "**Investigating the piecewise linearity and benchmark related to KOCZY-HIROTA fuzzy linear interpolation**", **Journal Scopus Indexed [Q3].***

*[94]. M. Alzubi and S. Kovacs, "**Some Considerations and a Benchmark Related to the CNF Property of the Koczy-Hirota Fuzzy Rule Interpolation**", **Journal Scopus Indexed [Q2].***

*[98]. Maen Alzubi, Mohammad Almseidin, Szilveszter Kovacs, and Mohd Aaqib Lone, "**Fuzzy Rule Interpolation Toolbox for the GNU Open-Source OCTAVE**", **Conference [IEEE].***

*[99]. M. Alzubi and S. Kovacs, "**Interpolative Fuzzy Reasoning Method Based on the Incircle of a Generalized Triangular Fuzzy Number**", **Journal Scopus Indexed [Q1].***

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