UNIVERSITY OF MISKOLC
Faculty of Mechanical Engineering and Informatics
Computer Science Department

Fuzzy Rule Interpolation and Benchmark System

Summary of Ph.D. dissertation

MA`EN MARWAN ALI AL-ZU`BI
MSC IN COMPUTER INFORMATION SYSTEM

'JÓZSEF HATVANY' DOCTORAL SCHOOL
OF INFORMATION SCIENCE, ENGINEERING AND TECHNOLOGY

ACADEMIC SUPERVISOR:
Dr. habil. Szilveszter Kovács

Miskolc, 2020

TABLE OF CONTENT

1. Introduction

FRI approaches are considerably useful for reasoning in case a sparse rule base. The FRI inference systems based on an interpolation concept that can generate a conclusion from existing rules. The FRI methods can divide into two groups. The first group produces the approximated conclusion from the observation directly. The second group produces the approximated conclusion from the observation base on two steps, (the first step, they interpolate a new rule that antecedent part at least overlaps the observation, the estimated conclusion determined in the second step based on the similarity between the observation and the antecedent part of the new rule). Besides that, a collection of conditions was suggested to be a baseline to compare and evaluate the performance of existing and upcoming FRI methods; still, most of the FRI methods not fulfilled some of the suggested conditions which could be useful important for the FRI concept. Because there is no common dataset that is suitable for comparing between FRI methods (e.g., benchmarks examples), all these reasons are leading us to the main aim of the research.

1.1. The Aim of Research

This research is divided into two goals:

- To generate the initial benchmark system (benchmark examples) related to the fuzzy set of the conclusion that must preserve a Piecewise Linearity (PWL) and must produce Convex and Normal Fuzzy (CNF). Hence, these benchmark examples could use to be a baseline for testing other FRI methods against situations that are not satisfied with the linearity and normality conditions for KH FRI.
- To develop a new method for fuzzy inference, which based on the Incircle of a triangular fuzzy number. This approach is suitable in case sparse fuzzy rule bases. It can handle the problems in some exists FRI methods and to be satisfied with most of the FRI conditions (such as normality, linearity, multi antecedent variables, approximation capability, extrapolation, Etc.).

2. Scientific Results

2.1. Interpolative Fuzzy Reasoning Method Based on the Incircle of a Generalized Triangular Fuzzy Number

A new technique was proposed in case a sparse fuzzy rule-based system called "Incircle FRI". This method based on the properties of the Incircle triangular fuzzy number, where the reference point and the fuzziness sides can be represented by the center point (GP) and sides (PS1, PS2, and PS3), respectively, as shown in **Fig. 1**. The proposed Incircle FRI follows geometrical considerations for performing fuzzy interpolation, taking care of producing Convex and Normal Fuzzy set (CNF).

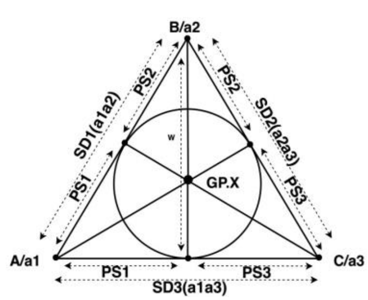


Fig. 1. *Notations of Triangular Fuzzy Number*

The Incircle FRI introduced to handle Triangular, Trapezoidal, Hexagonal, Multi antecedent variables, and Extrapolation as follows:

2.1.1. Single antecedent variable with Triangular fuzzy sets

Fig. 2 illustrates the suggested reference point of the fuzzy set indicated by ($GP_X.A$) of fuzzy set *A*, the main left, right and base sides of triangle *ABC* indicated by SD_1, SD_2, *and* SD_3, respectively, and the fuzziness sides, where PS_1 refers to the left fuzziness, and PS_3 refers to the right fuzziness.

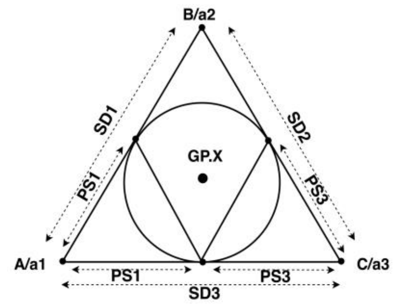


Fig. 2. *The Main Notations of the Triangular Fuzzy Number Represented by* $GPx, SD_1, SD_2, SD_3, PS_1, PS_2$ *and* PS_3

Fig. 3. Fuzzy Interpolative Reasoning Using Triangular Membership Functions

2.1.2. Single antecedent variable with Trapezoidal fuzzy sets

The Incircle concept of triangular fuzzy numbers can be extended to a trapezoidal fuzzy set. A trapezoidal fuzzy set can be represented by two triangular fuzzy sets $AL= (a_1, a_2, Mp; H)$ *and* $AR= (Mp, a_3, a_4; H)$, as shown in **Fig. 4.**

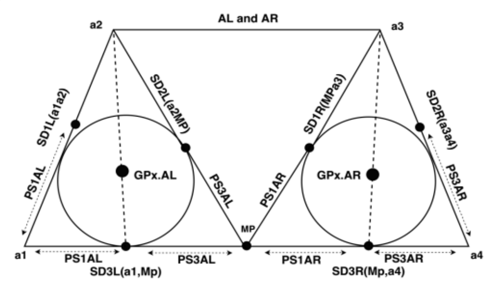


Fig. 4. *The Main Notations of the Trapezoidal Fuzzy Number Represented by Notations of Two Triangular Fuzzy Sets AL and AR.*

2.1.3. Extension of the Incircle FRI to single antecedent with hexagonal fuzzy set

A hexagonal fuzzy set based on the incircle FRI can be represented by two triangular fuzzy sets $AL= (a_1, a_2, a_3)$ and $AR= (a_4, a_5, a_6)$, as shown in **Fig. 5.**

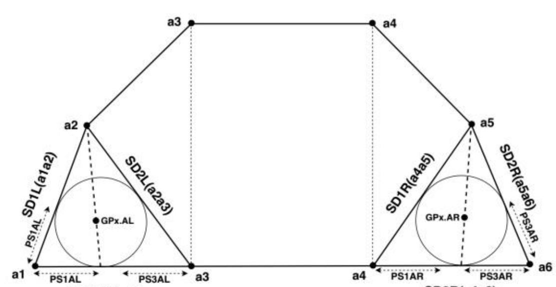


Fig. 5. *The Main Notations of the Hexagonal Fuzzy Number Represented by Notations of Two Triangular Fuzzy Sets AL and AR.*

2.1.4. Extension of the Incircle FRI to multiple fuzzy rules and multiple antecedent variables.

An example of the Incircle FRI reasoning using multiple fuzzy rules and multiple antecedents, which describe by trapezoidal fuzzy sets, as shown in **Fig. 6**. Then, the conclusion could be calculated based on the notations of the trapezoidal fuzzy set.

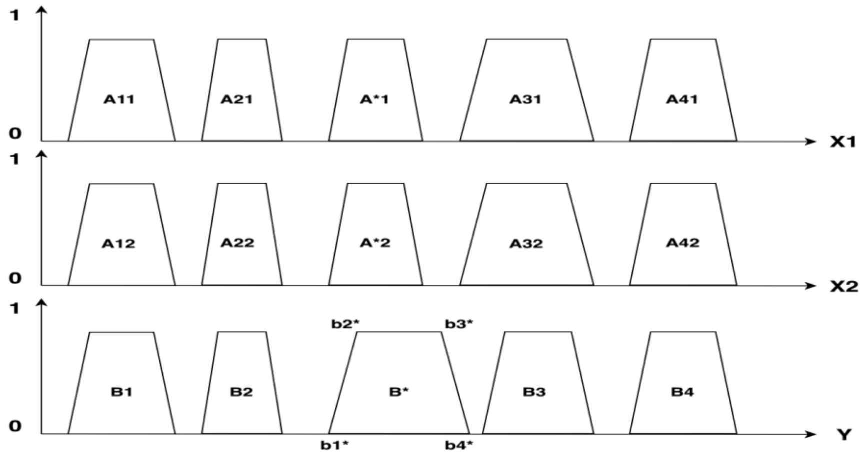


Fig. 6. *Fuzzy Interpolative Reasoning Using Trapezoidal Membership Functions*

2.1.5. Extension of the Incircle FRI to extrapolation using shift ratio and weight measurement

The Incircle FRI method is constructed to perform "Interpolation". Still, it cannot handle "Extrapolation" mainly due to two factors, which are the weight derivation and the lack of shift in a fuzzy set. The following subsubsection will debate the modification in weight derivation and introduce the shifting process.

2.1.5.1. Extended weight computation

The extended weight computation, an overall distance measure, was considered for the divisor as in given **Eq.(1)**. The denominator of the equation sums up all distances between the observation and rules. Normalization of the weights must be performed to each antecedent domain, as presented in **Eq.(2)**.

$$w_{ij} = 1 - \left| \frac{GPa_j^* - GPa_{ij}}{\sum_{i=1}^{n} \left| GPa_j^* - GPa_{ij} \right|} \right|, \tag{1}$$

$$W_{ij} = \frac{w_{ij}}{\sum_{i=1}^{n} w_{ij}}, \tag{2}$$

2.1.5.2. Extended Shift Ratio

The reference point (*GP*) of the rule sets can be successfully applied to interpolate the consequent in the case if only the two most adjacent rules are considered. By implementing the new weights, an interpolated observation can be calculated at the same (*GP*) as the real observation. This guarantees that the fuzzy conclusion is interpolated concerning the distance of (*GP*) of the observation and rules' antecedent. Nevertheless, when more than two rules are included, the intermediate observation, GpA' can be determined by **Eq.(3)**:

$$GPA'_j = \sum_{i=1}^{n} w_{ij} GPA_{ij}, \tag{3}$$

A shift of GpA' to GpA^* is required to align the intermediate observation to be the same reference point (GP) of the original observation. A shift ratio can be derived using the distance between (GP) of the observation and of the intermediate fuzzy set. The shift ratio δ_A can be calculated by **Eq.(4)**:

$$\delta_A = \frac{\sum_{j=1}^{m} \delta_{Aj}}{m}, \tag{4}$$

The reference point of the fuzzy conclusion, $GP_X.B^*$, is next calculated by the shift ratio δ_A together with the intermediate consequent as b' using the following:

$$GPx.B^* = GPx.B' + \delta_A(|GPx.B_n - GPx.B_1|). \tag{5}$$

2.1.6. Experiments and Discussion

This subsection describes the performance of the proposed "Incircle FRI" method, several experiments and comparisons were conducted of the Incircle FRI in comparison with different FRI methods, these experiments aim to prove the performance of the Incircle FRI with the triangular, trapezoidal, and hexagonal fuzzy sets, also for Extensions of the Incircle FRI Method in case multi antecedent variables, and to check the performance the capability of the Incircle FRI with extrapolation property.

2.1.6.1. Incircle FRI with Triangular fuzzy set

- **Example TR1** [7], [18], [20], [21], [22]:

Table 1. *Fuzzy Interpolative Reasoning Results of Example TR1*

Attribute Values	Methods	Results of Fuzzy Interpolative reasoning
A_1=[0 5 6] A_2=[11 13 14] B_1=[0 2 4] B_2=[10 11 13] A^*=[7 8 9]	KH FRI [1],[9],[13] KHstab FRI [3] VKK FRI [2] CCL FRI [18] HS FRI [7] HTY FRI [19] HCL FRI [15]	B^*=(6.36 5.38 7.38) B^*=(6.36 5.38 7.38) B^*=(6.15 5.38 7.84) B^*=(4.94 5.38 7.38) B^*=(5.83 6.26 7.38) B^*=(5.76 6.42 7.30) B^*=(6.36 6.58 7.38)
The Incircle FRI		**B^*=(4.95 5.42 7.16)**

- **Example TR2** [7], [18], [21]:

Table 2. *Fuzzy Interpolative Reasoning Results of Example TR2*

Attribute Values	Methods	Results of Fuzzy Interpolative reasoning
	KH FRI [1],[9],[13]	B^*=(7.27 5.38 6.25)
A_1=[0 5 6]	KHstab FRI [3]	B^*=(7.27 5.38 6.25)
A_2=[11 13 14]	VKK FRI [2]	B^*=(7.00 5.38 7.00)
B_1=[0 2 4]	CCL FRI [18]	B^*=(5.38 5.38 5.38)
B_2=[10 11 13]	HS FRI [7]	B^*=(6.49 6.49 6.49)
A^*=[8 8 8]	HTY FRI [19]	B^*=(6.49 6.49 6.49)
	HCL FRI [15]	B^*=(7.27 - 6.25)
The Incircle FRI		**B^*=(5.42 5.42 5.42)**

Note: the sign (-) indicates no clear evidence for the method to handle the case in the example

- **Example TR3** [7], [18], [21]:

Table 3. *Fuzzy Interpolative Reasoning Results of Example TR3*

Attribute Values	Methods	Results of Fuzzy Interpolative reasoning
	KH FRI [1],[9],[13]	B^*=(5.33 6.33 9.00)
A_1=[3 3 3]	KHstab FRI [3]	B^*=(5.33 6.33 9.00)
A_2=[12 12 12]	VKK FRI [2]	B^*=(- 0.00 -)
B_1=[4 4 4]	CCL FRI [18]	B^*=(5.33 6.33 8.33)
B_2=[10 11 13]	HS FRI [7]	B^*=(5.71 6.28 8.16)
A^*=[5 6 8]	HTY FRI [19]	B^*=(-)
	HCL FRI [15]	B^*=(5.33 6.55 9.00)
The Incircle FRI		**B^*=(5.28 6.37 8.28)**

Note: the sign (-) indicates no clear evidence for the method to handle the case in the example

2.1.6.2. Incircle FRI with Trapezoidal fuzzy set

- **Example TP1** [7], [18], [21]:

Table 4. *Fuzzy Interpolative Reasoning Results of Example TP1*

Attribute Values	Methods	Results of Fuzzy Interpolative reasoning
	KH FRI [1], [9], [13]	B^*=(5.45 4.25 7.50 8.5)
A_1=[0 4 5 6]	KHstab FRI [3]	B^*=(5.45 4.25 7.50 8.5)
A_2=[11 12 13 14]	VKK FRI [2]	B^*=(5.32 4.38 7.38 8.68)
B_1=[0 2 3 4]	CCL FRI [18]	B^*=(4.25 4.25 7.5 8.5)
B_2=[10 11 12 13]	HS FRI [7]	B^*=(5.23 5.23 7.61 8.32)
A^*=[6 6 9 10]	HTY FRI [19]	B^*=(4.98 7.44 6.44 8.06)
	HCL FRI [15]	B^*=(-)
The Incircle FRI		**B^*=(4.54 4.54 7.47 8.53)**

Note: the sign (-) indicates no clear evidence for the method to handle the case in the example

- **Example TP2** [18], [22]:

Table 5. *Fuzzy Interpolative Reasoning Results of Example TP2*

Attribute Values	Methods	Results of Fuzzy Interpolative reasoning
A_1=[0 5 6] A_2=[11 13 14] B_1=[0 2 3 4] B_2=[10 11 12 13] A^*=[7 8 9]	KH FRI [1], [9],[13] KHstab FRI [3] VKK FRI [2] CCL FRI [18] HS FRI [7] HTY FRI [19] HCL FRI [15]	B^*=(6.36 5.38 6.38 7.38) B^*=(6.36 5.38 6.38 7.38) B^*=(6.16 5.38 6.38 7.84) B^*=(4.94 5.38 6.38 7.38) B^*=(5.93 6.36 6.80 7.47) B^*=(5.87 6.20 7.20 7.41) B^*=(-)
The Incircle FRI		**B^*=(5.01 5.90 5.90 7.33)**

Note: the sign (-) indicates no clear evidence for the method to handle the case in the example

- **Example TP3** [18]:

Table 6. *Fuzzy Interpolative Reasoning Results of Example TP3*

Attribute Values	Methods	Results of Fuzzy Interpolative reasoning
A_1=[0 4 5 6] A_2=[11 12 13 14] B_1=[0 2 4] B_2=[10 11 13] A^*=[6 6 9 10]	KH FRI [1],[9],[13] KHstab FRI [3] VKK FRI [2] CCL FRI [18] HS FRI [7] HTY FRI [19] HCL FRI [15]	B^*=(5.45 4.25 6.50 8.50) B^*=(5.45 4.25 6.50 8.50) B^*=(5.31 5.38 5.38 8.68) B^*=(5.38 5.38 5.38 7.38) B^*=(5.46 5.46 5.46 8.55) B^*=(5.07 7.26 5.26 8.15) B^*=(-)
The Incircle FRI		**B^*=(4.37 4.37 6.66 8.57)**

Note: the sign (-) indicates no clear evidence for the method to handle the case in the example

2.1.6.3. Incircle FRI with Hexagonal fuzzy set

- **Example HEX1** [7], [21], [22]:

Table 7. *Fuzzy Interpolative Reasoning Results of Example HEX1*

Attribute Values	Methods	Results of Fuzzy Interpolative reasoning
A_1=[0 1 3 4 5 5.5] A_2=[11 11.5 12 13 13.5 14] B_1=[0 0.5 1 3 4 4.5] B_2=[10.5 11 12 13 13.5 14] A^*=[6 6.5 7 9 10 10.5]	KH FRI [1],[9],[13] KHstab FRI [3] VKK FRI [2] CCL FRI [18] HS FRI [7] HTY FRI [19] HCL FRI [15]	B^*=(5.73 5.74 5.75 5.89 7.25 8.56) B^*=(5.73 5.87 6.00 5.89 7.25 8.56) B^*=(5.60 5.69 5.71 6.26 7.46 8.68) B^*=(5.2 5.4 5.7 8.6 9.7 10.2) B^*=(5.47 5.79 6.08 8.42 9.23 9.70) B^*=(-) B^*=(-)
The Incircle FRI		**B^*=(5.59 6.02 6.55 8.63 9.52 10.13)**

Note: the sign (-) indicates no clear evidence for the method to handle the case in the example

2.1.6.4. Incircle FRI with extrapolation

- **Example Ext1:**

Table 8. *Fuzzy Interpolative Reasoning Results of Example Ext1*

Attribute Values	Methods	Results of Fuzzy Interpolative reasoning
A1=[3.5 5 7]; A2=[8 9 10] ; B1=[3 4 5]; B2=[9 10 11]; A_obs=[0 1 3];	KH FRI [1], [9],[13]	B^*=(-)
	KHstab FRI [3]	B^*=(4.82 6 6 7.18)
	VKK FRI [2]	B^*=(-)
	HTY FRI [19]	B^*=(-)
	HCL FRI [15]	B^*=(-)
	MACI FRI [4]	B^*=(-)
	IMUL FRI [6]	B^*=(-)
	CRF FRI [5]	B^*=(-)
The Incircle FRI		**B^*=(5.16 5.99 7.28)**

Note: the sign (-) indicates no clear evidence for the method to handle the case in the example

- **Example Ext2:**

Table 9. *Fuzzy Interpolative Reasoning Results of Example Ext2*

Attribute Values	Methods	Results of Fuzzy Interpolative reasoning
A_{11}=[8 9 10] A_{12}=[7 9 10] B_1=[9 10 11] A_{21}=[11 13 14] A_{22}=[11 12 13] B_2=[12 13 14] A^*_1=[3.5 5 7] A^*_2=[5 6 7]	KH FRI [1],[9],[13]	B^*=(-)
	KHstab FRI [3]	B^*=(10 11 11.94)
	VKK FRI [2]	B^*=(-)
	HTY FRI [19]	B^*=(-)
	HCL FRI [15]	B^*=(-)
	MACI FRI [4]	B^*=(-)
	IMUL FRI [6]	B^*=(-)
	CRF FRI [5]	B^*=(-)
The Incircle FRI		**B^*=(6.15 7.02 8.44)**

Note: the sign (-) indicates no clear evidence for the method to handle the case in the example

2.1.6.5. Incircle FRI with multiple fuzzy rules having multiple antecedents

- **Example MultiA1 [18]:**

Table 10. *Fuzzy Interpolative Reasoning Results of Example MultiA1*

Attribute Values	Methods	Results of Fuzzy Interpolative reasoning
A_{11}=[0 4 5 6] A_{12}=[1 2 3 4] B_1=[0 2 3 4] A_{21}=[11 12 13 14] A_{22}=[12 14 15 16] B_2=[10 11 12 13] A^*_1=[6 7 9 11] A^*_2=[6 8 10 12]	KH FRI [1],[9],[13]	B^*=(5 6.15 8.0 9.88)
	KHstab FRI [3]	B^*=(5 6.15 8.0 9.88)
	VKK FRI [2]	B^*=(4.9 6.11 8.12 9.88)
	CCL FRI [18]	B^*=(4.82 6.17 7.83 9.83)
	HS FRI [7]	B^*=(4.37 5.55 7.48 9.33)
	HTY FRI [19]	B^*=(-)
	HCL FRI [15]	B^*=(-)
	MACI FRI [4]	B^*=(5.32 6.87 7.37 7.87 8.87)
	IMUL FRI [6]	B^*=(4.3 6.6 7.6 10.68)
	CRF FRI [5]	B^*=(4.89 6.3 8.0 9.4)
The Incircle FRI		**B^*=(4.75 5.96 7.86 9.89)**

Note: the sign (-) indicates no clear evidence for the method to handle the case in the example

- **Example MultiA2 [52], [20]:**

Table 11. *Fuzzy Interpolative Reasoning Results of Example MultiA2*

Attribute Values Folder 1	Methods	Results of Fuzzy Interpolative reasoning
A_{11}=[0 1 1 3]	KH FRI [1],[9],[13]	B^*=(4.67 6.24 6.24 7.57)
A_{12}=[1 2 2 3]	KHstab FRI [3]	B^*=(6.21 7.66 7.66 8.9)
B_1=[0 2 2 3]	VKK FRI [2]	B^*=(4.87 6.26 6.26 7.53)
A_{21}=[8 9 9 10]	CCL FRI [18]	B^*=(4.79 6.22 6.22 7.54)
A_{22}=[7 9 9 10]	HS FRI [7]	B^*=(6.19 7.65 7.65 8.96)
B_2=[9 10 10 11]	HTY FRI [19]	B^*=(-)
A_{31}=[11 13 13 14]	HCL FRI [15]	B^*=(-)
A_{32}=[11 12 12 13]	MACI FRI [4]	B^*=(5.23 6.57 6.57 7.57)
B_3=[12 13 13 14]	IMUL FRI [6]	B^*=(4.1 6.25 6.25 8.89)
A^*_1=[3.5 5 5 7]	CRF FRI [5]	B^*=(5.39 6.25 6.25 7.25)
A^*_2=[5 6 6 7]		
The Incircle FRI		**B^*=(4.73 6.27 6.27 7.55)**

Note: the sign (-) indicates no clear evidence for the method to handle the case in the example

2.2. The Initial FRI Benchmark Examples Related to the CNF and PWL Properties of the Koczy-Hirota Fuzzy Rule Interpolation.

Many of the FRI methods suffer from the satisfaction of some FRI conditions related to the type of applicable linguistic terms and rule-base structure. For comparing the performance of different FRI conclusions, a proper benchmark system could be built by analyzing a set of conditions of the fuzzy sets, such as (core, boundary, slopes, etc.). The construction of such a benchmark system is not straightforward because of the numerous FRI methods and their special requirements.

One solution could be constructing an FRI benchmark system based on the common criteria of the FRI methods, according to fuzzy rule-base, fuzzy values, and observation configurations, where some of the required properties are not held in case of a given FRI method, e.g. the first method of the FRI "KH method". In the following, we will present all cases of the initial FRI benchmark, highlighting the problematic situations for KH FRI. Additionally, the benchmark will use as a baseline to compare and evaluate the performance of existing and upcoming FRI methods.

2.2.1. A FRI Benchmark Examples with Respect to the CNF Property of the Koczy-Hirota Fuzzy Rule Interpolation.

The main goal of this subsection is to take these boundary conditions and construct some benchmark examples to highlight the problematic properties of the original KH Fuzzy Rule Interpolation. Besides, this benchmark examples could be used for testing other FRI methods against these ill conditions. All benchmark examples introduced were implemented by the MATLAB FRI Toolbox [8], [10], which provides an easy-to-use framework for FRI applications.

2.2.1.1. The KH FRI CNF benchmark

The benchmark examples were constructed to highlight the conditions of the normality conclusion of the KH FRI. Various corollaries introduced to check the normality of the conclusion based on the core and (*Left-Right*) boundary lengths have a primary role in determining the normality. According to the prerequisites of the KH FRI, one-dimensional antecedents and consequents with trapezoidal, triangular, and singleton fuzzy sets, and two rules of the rule-bases could be considered. In the rest of the subsubsection, all the calculations and figures were prepared by the fuzzy rule interpolation (FRI) toolbox. We will present the cases where the conclusion of the KH FRI is normal and abnormal according to the equations and corollaries discussed previously.

The normality condition is always satisfied with the KH FRI if any of the following cases are met:

- **Case *CNF.C1*:** When the core and boundary lengths of the observation are greater or equal than the antecedent fuzzy sets $(KA^* >= KA)$, if $(Ka_i = KA)$, the normality of the KH FRI conclusion fuzzy set will always be satisfied. In this case, there is no restriction on the shape and size of the consequent (*KB*) (see Example "*CNF.KH_NOR.C1*").
- **Case *CNF.C2*:** When the core and boundary lengths of the fuzzy sets are the same $(KA = KB)$ *if* $(Ka_i = KA)$ *and* $(Kb_i = KB)$, the normality of the KH FRI conclusion fuzzy set is always satisfied. In this case, there is no restriction on the shape and size of the observation A^* (see Example "*CNF.KH_NOR.C2_1*").
- **Case *CNF.C3*:** If the core and boundary lengths of fuzzy sets $(KB > KA)$, where $(Kai = KA)$ and $(Kbi = KB)$, the conclusion of the KH FRI is always normality (see Examples "*CNF.KH_NOR.C3_1*" and "*CNF.KH_NOR.C3_2*").

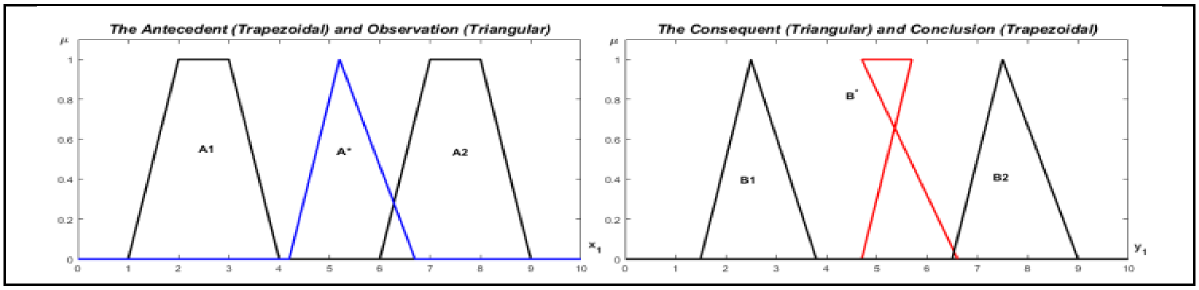
In contrast, the abnormality of the conclusion can appear in **Case *CNF.C4*** $(KB < KA)$. So, to demonstrate the abnormality problem, we will consider the length ratio between Ka^* and KB. Therefore, we will address the problem with different lengths of core and boundary.

Table 12 - **Table 15** describe the results notations to prove the normality of the KH FRI conclusion will not be satisfied. The Example "*CNF.KH_ABNOR.C4_1*" in **Table 12** shows the problem with the core length. Example "*CNF.KH_ABNOR.C4_2*" and Example "*CNF.KH_ABNOR.C4_3*" in **Table 13** and **Table 14** illustrate the problem of the left and right boundary. The Example "*CNF.KH_ABNOR.C4_4*" in **Table 15** shows the problem in both the core and boundary lengths.

Table 12 describes the abnormality in the core length of the KH FRI conclusion.

Table 12. *The Problem with Core Length, Abnormal Conclusion*

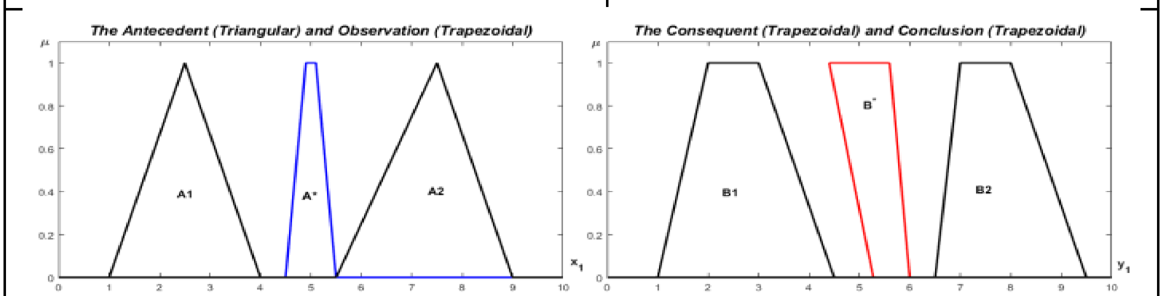
Example ("CNF.KH_ABNOR.C4_1")	
The values of the fuzzy sets: A_1= [1 2 3 4] A_2= [6 7 8 9] B_1= [1.5 2.5 2.5 3.8] B_2= [6.5 7.5 7.5 9] A^*= [4.2 5.2 5.2 6.7] B^*= [4.7 5.7 4.7 6.6]	**The case of the Core and (LF and RF) Boundary conclusion:** The length (LFBound) is (*<u>NORMAL</u>*) The length (Core) is (***<u>PROBLEM</u>***) The length (RFBound) is (*<u>NORMAL</u>*)
The length ratio between KA, Ka^* and KB: **LTBound**: *RatioLT1= 1, RatioLT2= 1.33* **Core**: *RatioC1= 1.25, RatioC2= 1* **RTBound**: *RatioRT1= 0.92, RatioRT2= 1.6*	**Notations length to determine the normality:** *LTBound1 = 0, LTBound2 = 5* *Core1 = 5, Core2 = 0* *RTBound1 = -9.25, RTBound2 = 17.28*



An explanation of the abnormality of the left boundary length in the KH FRI conclusion is shown in **Table 13**.

Table 13. *The Problem with Left Length, Abnormal Conclusion*

Example ("*CNF.KH_ABNOR.C4_2*")	
The values of the fuzzy sets: A_1= [1 2.5 2.5 4] A_2= [5.5 7.5 7.5 9] B_1= [1 2 3 4.5] B_2= [6.5 7 8 9.5] A^*= [4.5 4.9 5.1 5.5] B^*= [5.27 4.4 5.6 6.0]	**The case of the Core and (LF and RF) Boundary conclusion:** The length (LFBound) is (***PROBLEM***) The length (Core) is (*NORMAL*) The length (RFBound) is (*NORMAL*)
The length ratio between KA, Ka∗ and KB: ***LTBound***: *RatioLT1 = 1.5, RatioLT2 = 1.15* ***Core***: *RatioC1 = 0.8, RatioC2 = 1.04* ***RTBound***: *RatioRT1 = 1, RatioRT2 = 1.12*	**Notations length to determine the normality:** *LTBound1 = 30.15, LTBound2 = 6.80* *Core1 = -0.8, Core2 = 5.2* *RTBound1 = 3.85, RTBound2 = 5.85*



An illustration of the abnormality in the right boundary length in the KH FRI conclusion is displayed in **Table 14**.

Table 14. *The Problem with Right Length, Abnormal Conclusion*

Example ("*CNF.KH_ABNOR.C4_3*")	
The values of the fuzzy sets: A_1= [1.5 2.5 2.5 4.3] A_2= [6.5 7.5 7.5 8.8] B_1= [1 2 3 3.5] B_2= [6 7 8 8.9] A^*= [4.5 4.9 5.1 5.5] B^*= [4 4.4 5.6 4.94]	**The case of the Core and (LF and RF) Boundary conclusion:** The length (LFBound) is (*NORMAL*) The length (Core) is (*NORMAL*) The length (RFBound) is (***PROBLEM***)
The length ratio between KA, Ka∗ and KB: ***LTBound***: *RatioLT1 = 1, RatioLT2 = 1.11* ***Core***: *RatioC1 = 0.80, RatioC2 = 1.04* ***RTBound***: *RatioRT1 = 1.40, RatioRT2 = 1.14*	**Notations length to determine the normality:** *LTBound1 = 2.4, LTBound2 = 4.4* *Core1 = -0.8, Core2 = 5.2* *RTBound1 = 25.65, RTBound2 = 6.76*

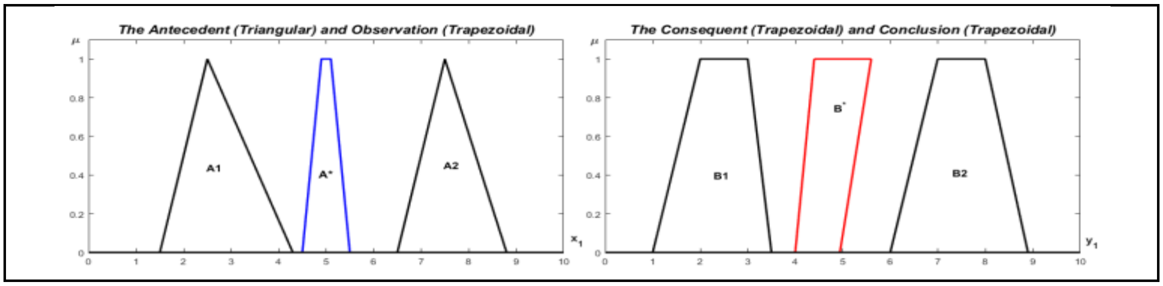
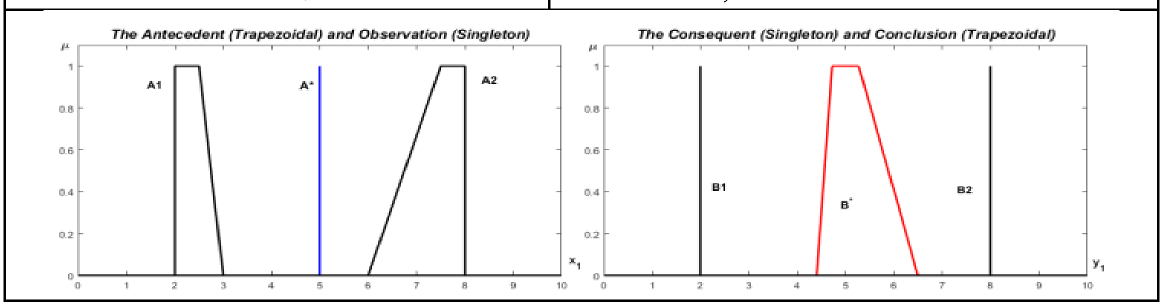


Table 15 describes the abnormality in both core and boundary lengths in the KH FRI conclusion.

Table 15. *The Problem with Core and Boundary Lengths, Abnormal Conclusion*

Example ("*CNF.KH_ABNOR.C4_4*")	
The values of the fuzzy sets: A_1=[2 2 2.5 3] A_2=[6 7.5 8 8] B_1=[2 2 2 2] B_2=[8 8 8 8] A^*=[5 5 5 5] B^*=[6.5 5.27 4.72 4.4]	**The case of the Core and (LF and RF) Boundary conclusion:** The length (LFBound) is (***PROBLEM***) The length (Core) is (***PROBLEM***) The length (RFBound) is (***PROBLEM***)
The length ratio between KA, Ka∗ and KB: ***LTBound***: *RatioLT1 = 1.5, RatioLT2 = 1* ***Core***: *RatioC1 = 1.2, RatioC2 = 1* ***RTBound***: *RatioRT1 = 1.2, RatioRT2 = 1*	**Notations length to determine the normality:** *LTBound1 = 27, LTBound2 = 0* *Core1 = 3, Core2 = 0* *RTBound1 = 9, RTBound2 = 0*
	

2.2.1.2. Discussion of the CNF benchmark examples

The cases and equations discussed earlier have been used to construct some benchmark examples. The constructed benchmark examples are based on Examples "*CNF.KH_ABNOR.C4_1*" – "*CNF.KH_ABNOR.C4_4*" that described the abnormality conclusion of the KH FRI, the abnormality could appear in case the corollary *(KB < KA), if (KA* < KB).*

Nevertheless, Examples "*CNF.KH_ABNOR.C4_1*" – "*CNF.KH_ABNOR.C4_4*", the abnormality conclusion could appear in case the length of the consequence (*KB*) is less than the length of the antecedents (*KA*) with taking into consideration the length ratio *(Ka* < KA)*, therefore, all notations used to check the normality conclusion are proved these examples not satisfied with CNF property as follows

In Example "*CNF.KH_ABNOR.C4_1*" as shown in Table 12, we can conclude the following:

- The core and boundary lengths of the fuzzy sets consequents (*KB*) are less than antecedents (*KA*), and the observation fuzzy set is less than consequent fuzzy sets *(KA* < KB)* (see case *CNF.C4*), then the conclusion is always abnormality.

- I.e., the core lengths of the consequents fuzzy sets are (*KB_1=0 (2.5-2.5)*, *KB_2=0 (7.5-7.5)*, and the average length of the consequents fuzzy sets *KB=0*), and the antecedents' fuzzy sets are (*KA_1=1 (3-2)*, *KA_2=1 (8-7)*, and the average length of the antecedent fuzzy sets *KA=1*), and the core length of the observation fuzzy set *KA^*=0 (5.2-5.2)*.

- For the length ratio notations used to prove the problem (abnormality in the core) as follows:
  - The length ratio of core is not satisfied, as (*RatioC1= 1.25*) exceeds the (*RatioC2= 1*).
- From another side, an issue for core length.
  - The core length values, (*Core1= 5*) is greater than (*Core2 = 0*).
- In this case *(KB<KA) and (KA^* < KB)*, the conclusion *B^*= [4.7 5.7 4.7 6.6]* is always abnormality, as shown by values of the core (5.7 is greater than 4.7).

In Example "*CNF.KH_ABNOR.C4_2*" as shown in Table 13, we can conclude the following:

- The core and boundary lengths of the fuzzy sets *KB* < *KA* and *KA^** < *KB*. In this case, the conclusion *B^*= [5.27 4.4 5.6 6.0]* is always abnormality, where the problem could appear in the left values as (5.27 is greater than 4.4).
  - I.e., the left lengths of the consequents fuzzy sets are (*KB_1=1 (2-1)*, *KB_2=0.5 (7-6.5)*, and the average length of the consequents fuzzy sets *KB=0.75*), and the antecedents fuzzy sets are (*KA_1=1.5 (2.5-1)*, *KA_2=2 (7.5-5.5)*, and the average length of the antecedent fuzzy sets *KA=1.75*), and the left length of the observation fuzzy set *KA^*=0.4 (4.9-4.5)*.
- For the length ratio notations used to prove the problem (abnormality in the core) as follows:
  - The length ratio of the left boundary is not satisfied, where (*RatioLT1= 1.5*) exceeds the (*RatioLT2= 1.15*).
- Also, a problem for core length is demonstrated:
  - The left length values, where (*LTBound1= 30.15*) is greater than (*LTBound2 = 6.80*), and therefore, the conclusion is suffering from the abnormality.

In Example "*CNF.KH_ABNOR.C4_3*" as shown in Table 14, we can conclude the following:

- The core and boundary lengths of *KB* < *KA* and *KA^** < *KA*. In this case, the conclusion *B^*= [4 4.4 5.6 4.94]* is always abnormality, where the problem could appear in the right values (5.6 is greater than 4.94).
- I.e., the right lengths of the consequents fuzzy sets are (*KB_1=0.5 (3.5-3)*, *KB_2=0.9 (8.9-8)*, and the average length of the consequents fuzzy sets *KB=0.7*), and the antecedents fuzzy sets are (*KA_1=1.8 (4.3-2.5)*, *KA_2=1.3 (8.8-7.5)*, and the average length of the antecedent fuzzy sets *KA=1.55*), and the right length of the observation fuzzy set *KA^*=0.4 (5.5-5.1)*.
- For the length ratio notations used to prove the problem (abnormality in the core) as follows:
  - The length ratio of the right boundary is not satisfied, where (*RatioRT1= 1.40*) exceeds (*RatioRT2= 1.14*).
- Also, a problem for core length is demonstrated:
  - The left length values, where (*RTBound1= 25.65*) is greater than (*RTBound1= 6.76*), and therefore, the conclusion is suffering from the abnormality.

In Example "*CNF.KH_ABNOR.C4_4*" as shown in Table 15, we can conclude the following:

- We can see the problems in both of the core and boundary lengths, where the conclusion B^{*}= *[6.5 5.27 4.72 4.4]* is always abnormality.
- Regarding the notations of the core and boundaries lengths are not satisfied with the normality:
  - Left Boundary length**:** *LTBound1=27 > LTBound2=0*,
  - Core length**:** *Core1=3 > Core2=0*,
  - Right Boundary length**:** *RTBound1=9 > RTBound2=0*.
- In addition, the lengths ratios of the core and boundaries are not satisfied because (Ratio1) is greater than (Ratio2) for the core and boundary of the conclusion:
  - Left ratio: *RatioLT1=1.5 > RatioLT2=1*
  - Core: *RatioC1=1.2 > RatioC2=1*
  - Right ratio: *RatioRT1=1.2 > RatioRT2=1*

2.2.2. A FRI Benchmark Example with Respect to the PWL Property of the Koczy-Hirota Interpolation

This subsection aims to highlight the problematic properties of the KH FRI method to prove its efficiency with PWL condition in order to construct benchmark examples. This benchmark can serve as a baseline for testing other FRI methods against cases that the KH FRI is not satisfied with the linearity condition. All benchmark examples were constructed using notations and equations detailed in [14], [16], [17].

2.2.2.1. The KH FRI PWL benchmark

Benchmark examples are constructed to demonstrate the validity of the PWL condition of the KH FRI method. Different equations were used to check the linearity of the KH FRI conclusion, and also to construct the benchmark examples. The left and right slopes of the fuzzy rule and observation play a significant role in preserving the conclusion's linearity.

The benchmark examples constructed using one-dimensional input and output variables, the triangular membership function, and two fuzzy rules represent fuzzy sets of the antecedent, consequent, and observation. Benchmark examples are divided into two groups. The first group presents the rule-base, observation configurations, where the KH FRI conclusion is satisfied with the PWL condition. The second group shows the examples where the conclusions of the KH FRI are not satisfied with the PWL condition. We will present the cases related to the PWL property of the KH FRI conclusion.

The KH FRI conclusion is always satisfied with PWL condition if the following cases are met:

- **For Case *PWLC1*:** When the left and right slopes A_i and B_i fuzzy sets are identical *(e.g., for left slop* $a_{12} - a_{11} = a_{22} - a_{21}$ *and* $b_{12} - b_{11} = b_{22} - b_{21}$*)*. The conclusion of KH FRI will always be satisfied with the linearity condition (see the notations of the Example "*PWL.LIN.C1*").
- **For Case *PWLC2*:** If two adjacent fuzzy rule bases $A_1 \rightarrow B_1$ *and* $A_2 \rightarrow B_2$ *(e.g., for left slop: Rule1* $(a_{12} - a_{11} = b_{12} - b_{11})$*, Rule2* $(a_{22} - a_{21} = b_{22} - b_{21})$ have the same left and right slopes

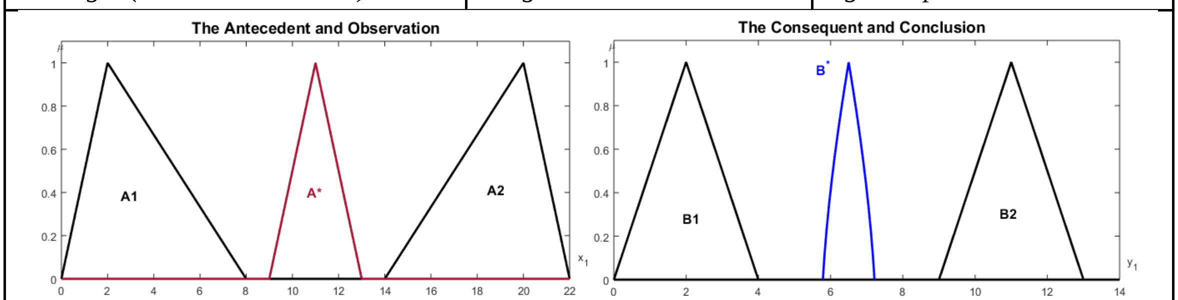
and the same characteristic points on the universe of discourse. Then, the KH FRI conclusion will always be satisfied with the linearity condition (see the Example "*PWL.LIN.C2*").

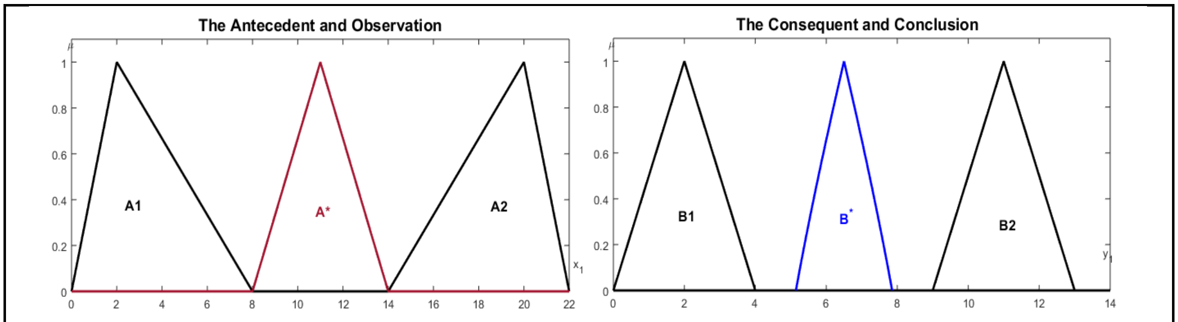
- **For Case *PWLC3*:** When the fuzzy sets of the antecedents A_i and the observation A^* have the same left and right slopes PWL. Then, the conclusion of the KH FRI will always be satisfied with the linearity condition (see the Example "*PWL.LIN.C3*").
- **For Case *PWLC4*:** When the left and right slopes for all fuzzy sets of two adjacent fuzzy rule bases and observation are equidistant $(A_i = B_i = A^*)$. Therefore, the conclusion of the KH FRI will always be satisfied with the linearity condition (see the notations of the Example "*PWL.LIN.C4*").

The conclusions of the KH FRI are not satisfied with PWL condition if the following cases hold.

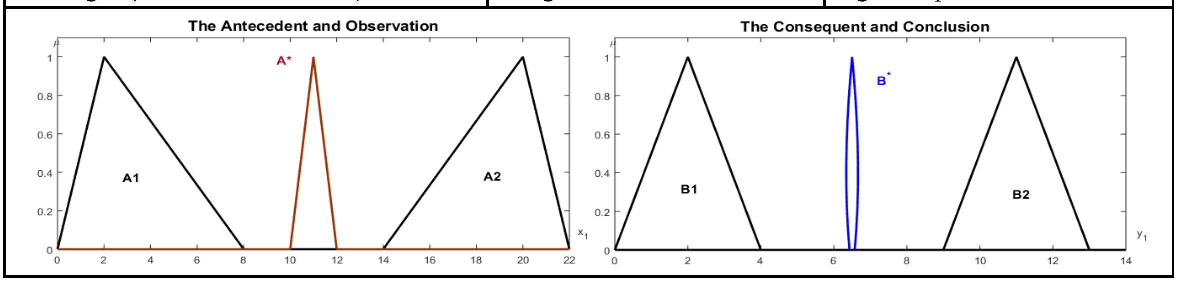
- **According to Case *PWLC1*:** When the left and right slopes A_i and B_i are incompatible *(e.g., for left slop* $(a_{12} - a_{11} \neq a_{22} - a_{21})$ *and* $(b_{12} - b_{11} = b_{22} - b_{21})$ whereas $A_i \neq A^*$, in this case, the linearity conclusion of KH FRI is not satisfied. Example "*PWL.NONLIN.C1*" constructed to prove the problem, which will be described by three different situations based on the characteristic points of the observation A^* to compare its linearity conclusions. **Table 16** illustrates notations that describe the problem according to the three situations.

Table 16. *The Problem with Slopes to Case PWLC1 Which is not Preserving PWL*

Example "*PWL.NONLIN.C1*" situation1 when e.g. the left slope (b1(2) - b1(1) = b2(2) - b2(1)) = A^*		
The characteristic points of the fuzzy sets: A_1= [0 2 2 8] A_2= [14 20 20 22] A^*= [9 11 11 13] B_1= [0 2 2 4] B_2= [9 11 11 13] B^*= [5.79 6.50 6.50 7.21]	**The length of left and right slopes of the fuzzy sets:** ***For left:*** $A_1=2, A_2=6, A^*=2, B_1=2, B_2=2$ ***For Right:*** $A_1=6, A_2=2, A^*=2, B_1=2, B_2=2$	
ΔB^* *Left (Maximum-deviation) = 0.08* ΔB^* *Right (Maximum-deviation) = 0.08*	*E.Left = 1.2857* *E.Right = 1.2857*	*Left.Slope = 0* *Right.Slope = 0*
		
Example "*PWL.NONLIN.C1*" situation2 when e.g. the left slope (b1(2) - b1(1) = b2(2) - b2(1)) < A^*		
The characteristic points of the fuzzy sets: A_1= [0 2 2 8] A_2= [14 20 20 22] A^*= [8 11 11 14] B_1= [0 2 2 4] B_2= [9 11 11 13] B^*= [5.14 6.50 6.50 7.86]	**The length of left and right slopes of the fuzzy sets:** ***For left:*** $A_1=2, A_2=6, A^*=3, B_1=2, B_2=2$ ***For Right:*** $A_1=6, A_2=2, A^*=3, B_1=2, B_2=2$	
ΔB^* *Left (Maximum-deviation) = 0.04* ΔB^* *Right (Maximum-deviation) = 0.04*	*E.Left = 0.6429* *E.Right = 0.6429*	*Left.Slope = 0* *Right.Slope = 0*



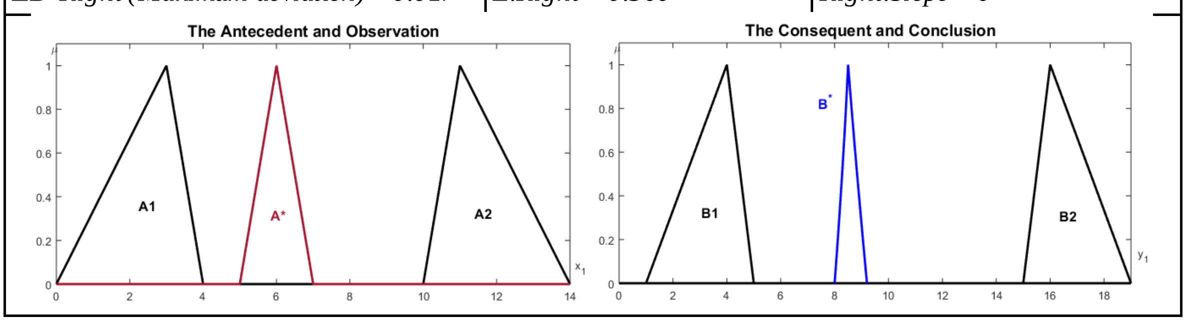
Example "*PWL.NONLIN.C1*" situation3 when e.g. the left slope (b1(2) - b1(1) = b2(2) - b2(1)) > A^*		
The characteristic points of the fuzzy sets: A_1= [0 2 2 8] A_2= [14 20 20 22] A^*= [10 11 11 12] B_1= [0 2 2 4] B_2= [9 11 11 13] B^*= [6.4286 6.5 6.5 6.571]	**The length of left and right slopes of the fuzzy sets:** ***For left***: $A_1=2, A_2=6, A^*=1, B_1=2, B_2=2$ ***For Right***: $A_1=6, A_2=2, A^*=1, B_1=2, B_2=2$	
ΔB^* *Left (Maximum-deviation) = 0.121* ΔB^* *Right (Maximum-deviation) = 0.121*	*E.Left = 1.9286* *E.Right = 1.9286*	*Left.Slope = 0* *Right.Slope = 0*



- **About Case *PWLC2*:** When the two adjacent fuzzy rule bases $A_1 \rightarrow B_1$ and $A_2 \rightarrow B_2$ have the same left and right slopes but have different characteristic points on the universe of discourse, in this case, the linearity conclusion of KH FRI is not satisfied. Example "*PWL.NONLIN.C2*" constructed to prove the issue, as shown in **Table 17**.

Table 17. *The Problem with Slopes to Case PWLC2 Which is not Preserving PWL*

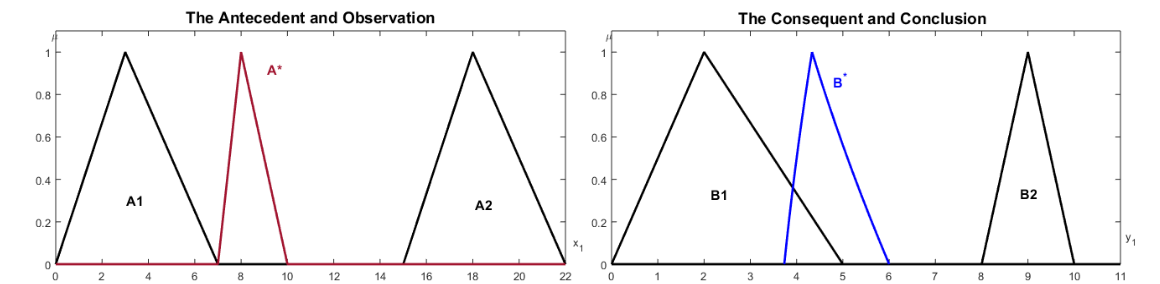
Example "*PWL.NONLIN.C2*"		
The characteristic points of the fuzzy sets: A_1= [0 3 3 4] A_2= [10 11 11 14] A^*= [5 6 6 7] B_1= [1 4 4 5] B_2= [15 16 16 19] B^*= [8 8.5 8.5 9.2]	**The length of left and right slopes of the fuzzy sets:** ***For left***: $A_1=3, A_2=1, A^*=1, B_1=3, B_2=1$ ***For Right***: $A_1=1, A_2=3, A^*=1, B_1=1, B_2=3$	
ΔB^* *Left (Maximum-deviation) = 0.028* ΔB^* *Right (Maximum-deviation) = 0.017*	*E.Left = 0.500* *E.Right = 0.300*	*Left.Slope = 0* *Right.Slope = 0*



- **Referring to Case *PWLC3*:** When the left and right slopes of the antecedents A_i $(a_{12} - a_{11} = a_{22} - a_{21})$ and the observation A^* are not equivalent, whereas $A_i \neq B_i$, then, the linearity conclusion of KH FRI is not satisfied. Refer to Corollary *PWLCR1*, Example "*PWL.NONLIN.C3*" is applied the polynomial condition when $(a_{12} - a_{11} = a_{22} - a_{21})$; however, it is not linear. **Table 18** describes notations which prove the problem to this case.

Table 18. *The Problem with Slopes to Case PWLC3 Which is not Preserving PWL*

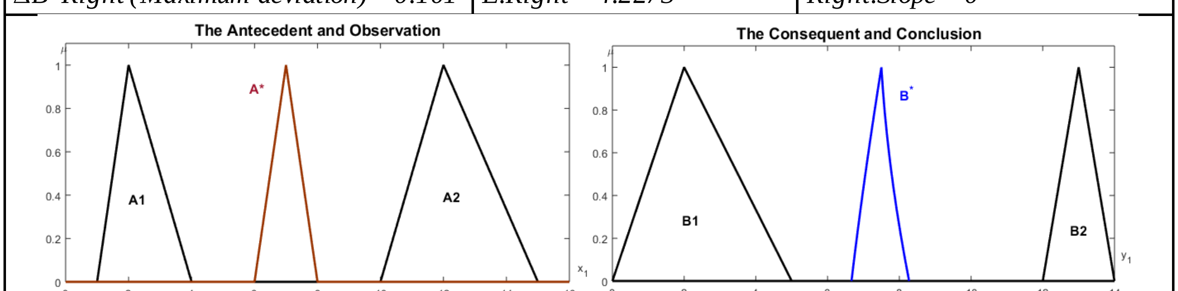
Example "*PWL.NONLIN.C3*"		
The characteristic points of the fuzzy sets: A_1= [0 3 3 7] A_2= [15 18 18 22] A^*= [7 8 8 10] B_1= [0 2 2 5] B_2= [8 9 9 10] B^*= [3.73 4.33 4.33 6.0]	**The length of left and right slopes of the fuzzy sets:** ***For left:*** $A_1=3, A_2=3, A^*=1, B_1=2, B_2=1$ ***For Right:*** $A_1=4, A_2=4, A^*=2, B_1=3, B_2=1$	
ΔB^* *Left (Maximum-deviation) = 0.033* ΔB^* *Right (Maximum-deviation) = 0.067*	*E.Left = NAN* *E.Right = NAN*	*Left.Slope = 0* *Right.Slope = 0*



- **According to Case *PWLC4*:** When values of the left and right slopes of fuzzy rule bases and observation are not similar $(A_i \neq B_i \neq A^*)$, in this case, the linearity conclusion of KH FRI is not satisfied. Example "*PWL.NONLIN.C4*" created to demonstrate the problem, as shown in **Table 19**.

Table 19. *The Problem with Slopes to Case PWLC4 Which is not Preserving PWL*

Example "*PWL.NONLIN.C4*"		
The characteristic points of the fuzzy sets: A_1= [1 2 2 4] A_2= [10 12 12 15] A^*= [6 7 7 8] B_1= [0 2 2 5] B_2= [12 13 13 14] B^*= [6.67 7.5 7.5 8.27]	**The length of left and right slopes of the fuzzy sets:** ***For left:*** $A_1=1, A_2=2, A^*=1, B_1=2, B_2=1$ ***For Right:*** $A_1=2, A_2=3, A^*=1, B_1=3, B_2=1$	
ΔB^* *Left (Maximum-deviation) = 0.031* ΔB^* *Right (Maximum-deviation) = 0.101*	*E.Left = 1.1667* *E.Right = 4.2273*	*Left.Slope = 0* *Right.Slope = 0*



2.2.2.2. Discussion of the KH FRI PWL benchmark examples

Several cases and equations have been used to construct some benchmark examples. The constructed benchmark examples are based on Examples *"PWL.NONLIN.C1" – "PWL.NONLIN.C4"* describe cases, where the conclusions of the KH FRI are not satisfied with PWL. These examples have been presented based on two facts, either if the conclusion is close to linearity (Example *"PWL.NONLIN.C1"* situation2) or far from linearity (Example "PWL.NONLIN.C1" situations3). Regarding the estimated error notations will not be satisfied with the PWL condition because the value is always 0 for (left.Slope and right.Slope). Another notation will be discussed in detail as follows:

According to Example "*PWL.NONLIN.C1*" as shown in Table 16, we can conclude the following:

- There are three different situations based on the characteristic points of the observation as:
  - Situation1 *(when* $(b_{12} - b_{11} = b_{22} - b_{21}) = A^*$*),*
  - Situation2 *(when* $(b_{12} - b_{11} = b_{22} - b_{21}) < A^*$*),*
  - Situation3 *(when* $(b_{12} - b_{11} = b_{22} - b_{21}) > A^*$).
- **Fig. 7** explains the difference between real and linear approximation functions for each situation, the maximum deviation for left and right slopes in situation1 is *0.08*, and situation2 is smaller than situation1, which is *0.04*, in contrast, situation3 has the high deviation is *0.121*.

α - Levels		0.000	0.100	0.200	0.300	0.400	0.500	0.600	0.700	0.800	0.900	1.000
ΔB* Left Slope	Situation 1	0.000	0.032	0.056	0.071	0.079	0.080	0.075	0.064	0.048	0.026	0.000
	Situation 2	0.000	0.016	0.028	0.036	0.040	0.040	0.038	0.032	0.024	0.013	0.000
	Situation 3	0.000	0.048	0.083	0.107	0.119	0.121	0.113	0.096	0.072	0.039	0.000
ΔB* Right Slope	Situation 1	0.000	0.026	0.048	0.064	0.075	0.080	0.079	0.071	0.056	0.032	0.000
	Situation 2	0.000	0.013	0.024	0.032	0.038	0.040	0.040	0.036	0.028	0.016	0.000
	Situation 3	0.000	0.039	0.072	0.096	0.113	0.121	0.119	0.107	0.083	0.048	0.000

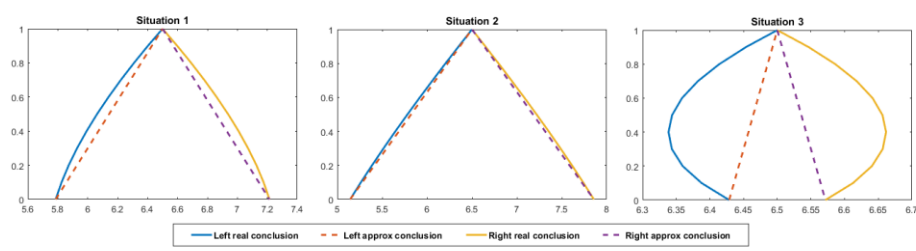


Fig. 7. *The Difference Between the Linear Approximation and Real Functions of the Left and Right Slopes for* $\alpha \in [0,1]$ *To Example "PWL.NONLIN.C1"*

- On the other hand, the error ratios are different for three situations, situation3 has a large error ratio compared to situation1 and situation2, where the error ratio of the left and right slopes of (situation3 is 1.9286), (situation1 is *1.2857)*, and (situation2 is *0.6429)*. Then, situation3 is far from linearity, in contrast to situation2, which is closer than situation1 to linearity.
- The conclusion is always not satisfied with the PWL condition, where the value is always 0 for (left.Slope and right.Slope).

In Example "*PWL.NONLIN.C2*" as shown in Table 17, we can conclude the following:

- The problem appears when the left and right slopes of fuzzy rule bases are the same, but the characteristic points of the fuzzy sets of A_i and B_i are different on the universe of discourse. In this case, the conclusion KH FRI is not satisfied with linearity. E.g.:
  - The Rule $A_1 \Rightarrow B_1$ (for the lower $A_1=3$ *and* $B_1=3$), and (for the upper $A_1=1$ *and* $B_1=1$).
  - The Rule $A_2 \Rightarrow B_2$ (for the lower $A_1=1$ *and* $B_1=1$), and (for the upper $A_1=3$ *and* $B_1=3$).
  - The characteristic points $A_1= [0\ 3\ 3\ 4] \Rightarrow B_1= [1\ 4\ 4\ 5]$ *and* $A_2= [10\ 11\ 11\ 14] \Rightarrow B_2= [15\ 16\ 16\ 19]$.
- The deviation for the left slope is greater than the right slope, where the left slope is *0.028*, and the right slope is *0.017*, as shown in **Fig. 8**.

α - Levels	ΔB*	
	Left Slope	Right Slope
0.000	0.000	0.000
0.100	0.009	0.007
0.200	0.017	0.011
0.300	0.022	0.015
0.400	0.026	0.016
0.500	0.028	0.017
0.600	0.027	0.016
0.700	0.024	0.013
0.800	0.019	0.010
0.900	0.011	0.006
1.000	0.000	0.000

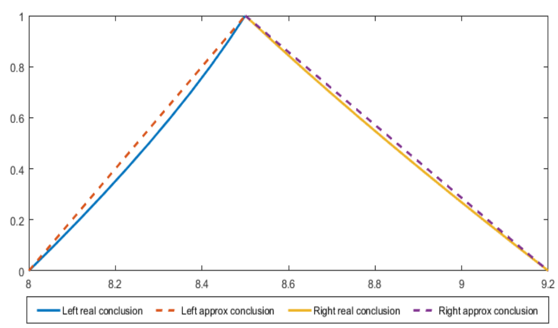


Fig. 8. *The Difference Between the Linear Approximation and Real Functions of the Left and Right Slopes for α ∈[0,1] To Example "PWL.NONLIN.C2"*

- Also, the error ratio of the left slope is *0.5* is far from linearity to the right slope is *0.3*.
- The conclusion is always not satisfied with the PWL condition, where the value is always 0 for (left.Slope and right.Slope).

In Example "*PWL.NONLIN.C3*" as shown in Table 18, we can conclude the following:

- It demonstrates the conclusion of KH FRI is not satisfied with linearity according to the results of equations and notations. When the left and right slopes of the antecedents A_i $(a_{12} - a_{11} = a_{22} - a_{21})$ and the observation A^* are not equivalent whereas $A_i \neq B_i$, then,
- **Fig. 9** explains the difference between real and linear approximation functions where the maximum deviation for left and right slopes are *0.033* and *0.067*, respectively.

α - Levels	ΔB*	
	Left Slope	Right Slope
0.000	0.000	0.000
0.100	0.012	0.024
0.200	0.021	0.043
0.300	0.028	0.056
0.400	0.032	0.064
0.500	0.033	0.067
0.600	0.032	0.064
0.700	0.028	0.056
0.800	0.021	0.043
0.900	0.012	0.024
1.000	0.000	0.000

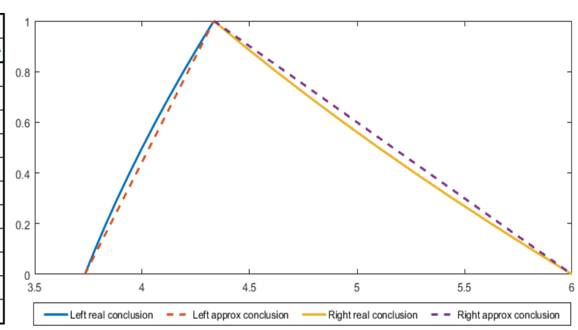


Fig. 9. *The Difference Between the Linear Approximation and Real Functions of the Left and Right Slopes for α ∈[0,1] To Example "PWL.NONLIN.C3"*

- The error ratio for the left and the right slopes are *NAN*, by referring to Corollary *PWLCR1*, this example has achieved the condition of polynomiality because the left and right slopes of A_1 and A_2 are similar, but not linear.
- The conclusion is always not satisfied with the PWL condition because the value of this equation is always 0 for (left.Slope and right.Slope).

Also, in Example "*PWL.NONLIN.C4*" as shown in Table 19, we can conclude the following:

- All fuzzy sets of fuzzy rule bases and observation are different $(A_i \neq B_i \neq A^*)$, then, the conclusion KH FRI is also not satisfied with the linearity condition.
- Additionally, **Fig. 10** defines the difference between real and linear approximation functions as follows:

α - Levels	ΔB*	
	Left Slope	Right Slope
0.000	0.000	0.000
0.100	0.409	0.207
0.200	0.535	0.365
0.300	0.556	0.475
0.400	0.525	0.538
0.500	0.467	0.556
0.600	0.390	0.529
0.700	0.303	0.459
0.800	0.207	0.347
0.900	0.106	0.193
1.000	0.000	0.000

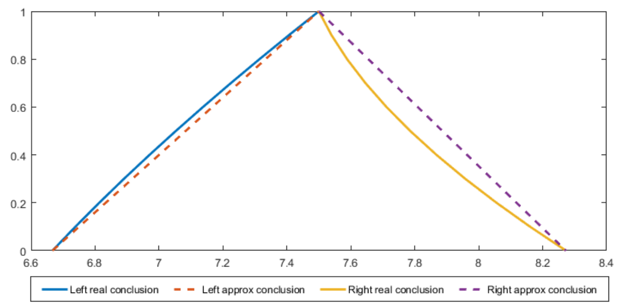


Fig. 10. *The Difference Between the Linear Approximation and Real Functions of the Left and Right Slopes for α ∈ [0,1] To Example "PWL.NONLIN.C4"*

- The error ratio of the linearity in the right slope is *4.2273*, which is so far than in the left slope is *1.1667*.
- The conclusion is always not satisfied with PWL condition because the value is always 0 for (left.Slope and right.Slope).

2.2.3. The Application of the CNF and PWL Benchmark Examples

This subsection will discuss the efficiency of the CNF and PWL benchmarks in comparing some of the FRI methods implemented via FRI Toolbox. Besides, the validity of the proposed Incircle FRI method will be discussed based on this benchmark and compared it to other FRI methods.

2.2.3.1. Testing FRI methods based on the CNF benchmark examples

In the following, some of the FRI methods (KHstabilized [3], MACI [4], VKK [2], and CRF [5]) will be tested and compared according to the CNF benchmark examples. To offer a simple way of comparison, we focus on the cases that demonstrated abnormality conclusion (Examples: "*CNF.KH_ABNOR.C4_1*" – "*CNF.KH_ABNOR.C4_4*"). This comparison shows the difference between the results of the selected methods according to the CNF property.

Table 20 illustrates the results of the selected FRI methods (KH FRI [1], [9], [13], KHstabilized FRI [3], MACI FRI [4], VKK FRI [2], and CRF FRI [5]), as shown by the values of the conclusions B^*.

Table 20. *The Conclusions of the Selected Methods (KH, KHstabilized, MACI, VKK, and CRF) Related to Examples "CNF.KH_ABNOR.C4_1 "– "CNF.KH_ABNOR.C4_4 "*

Method	**Approximate Conclusion B***			
	Example ***CNF.KH_ABNOR.C4_1***	**Example** ***CNF.KH_ABNOR.C4_2***	**Example** ***CNF.KH_ABNOR.C4_3***	**Example** ***CNF.KH_ABNOR.C4_4***
KH FRI [1],[9],[13]	Abnormality [4.7 5.7 4.7 6.6]	Abnormality [5.27 4.4 5.6 6.0]	Abnormality [4 4.4 5.6 4.94]	Abnormality [6.5 5.27 4.72 4.4]
KHstab FRI [3]	Abnormality [4.7 5.7 4.7 6.6]	Abnormality [5.27 4.4 5.6 6.0]	Abnormality [4 4.4 5.6 4.94]	Abnormality [6.5 5.27 4.72 4.4]
MACI FRI [4]	Normal [4.2 5.2 5.2 6.6]	Normal [3.89 4.5 5 5 5.5 7]	Normal [3.5 4.5 5 5 5.5 6.1]	Normal [5 5 5 5]
VKK FRI [2]	Normal [4.6 5.2 5.2 6.66]	Abnormality [out range]	Abnormality [out range]	Abnormality [5.3 5 5 5.3]
CRF FRI [5]	Normal [3.9 5.25 5.25 6.75]	Normal [4.5 4.9 5.0 5.1]	Normal [4.8 4.9 5.0 5.4]	Normal [5 5 5 5]

2.2.3.2. Testing the Incircle FRI method based on the CNF property

The proposed Incircle FRI method is tested to prove the validity of the conclusions according to the CNF property. Firstly, Incircle FRI was tested according to the CNF Benchmark. Secondly, Incircle FRI was tested according to all examples used in the dissertation. Therefore, the Incircle FRI method is a suitable approach to be implemented as an inference system because its conclusions succeeded with CNF property to all examples. **Table 21** and **Table 22** describe the validity conclusions of the Incircle FRI method with CNF property, as shown by values of the conclusions B^*.

Table 21. *The Conclusions of the Incircle FRI Method Related to CNF Benchmark Examples ("CNF.KH_ABNOR.C4_1 "– "CNF.KH_ABNOR.C4_4 ")*

Approximate Conclusion B^* of the Incircle FRI method			
Example *"CNF.KH_ABNOR.C4_1"*	Example *"CNF.KH_ABNOR.C4_2 "*	Example *"CNF.KH_ABNOR.C4_3 "*	Example *"CNF.KH_ABNOR.C4_4 "*
Normal [4.82 5.55 5.55 6.49]	Normal [4.36 4.86 5.29 5.87]	Normal [4.10 4.80 5.23 5.65]	Normal [5.03 5.023 5.03 5.03]

Table 22. *The Conclusions of the Incircle FRI Method Related to Examples CNFIncircle(TR1) – CNFIncircle(Diff4)*

Approximate Conclusion B^* of the Incircle FRI method			
Example "*CNFIncircle(TR1)*"	Example "*CNFIncircle(TR2)*"	Example "*CNFIncircle(TR3)*"	
Normal [4.95 5.42 7.16]	Normal [5.42 5.42 5.42]	Normal [5.28 6.37 8.28]	
Example "*CNFIncircle(TP1)*"	Example "*CNFIncircle(TP2)*"	Example "*CNFIncircle(TP3)*"	
Normal [4.54 4.54 7.47 8.53]	Normal [5.01 5.90 5.90 7.33]	Normal [4.37 4.37 6.66 8.57]	
Example "*CNFIncircle(Diff1)*"	Example "*CNFIncircle(Diff2)*"	Example "*CNFIncircle(Diff3)*"	Example "*CNFIncircle(Diff4)*"
Normal [5.07 5.77 5.77 6.93]	Normal [2.07 3.0 3.00 3.93]	Normal [5.74 5.74 8.26 8.26]	Normal [5.13 5.13 5.13 5.13]

2.2.3.3. Testing FRI methods based on the PWL benchmark examples

The FRI methods (KHstabilized FRI [3], VKK FRI [2], FRIPOC FRI [27], VEIN FRI [23], and Incircle FRI) were compared according to the PWL benchmark examples. To offer a simple way of comparison we focused on the cases that the KH FRI method demonstrated the fails of preserving PWL, which was represented by Examples ("PWL.NONLIN.C1" situations1,2,3, "PWL.NONLIN.C2", "PWL.NONLIN.C3" and "PWL.NONLIN.C4"). Multiple α levels were computed to perform the comparisons. According to the results of the FRI methods (KHstabilized FRI [3], VKK FRI [2], FRIPOC FRI [27], VEIN FRI [23], and Incircle FRI) to the PWL benchmark examples ("*PWL.NONLIN.C1*" situations1,2,3, "*PWL.NONLIN.C2*", "*PWL.NONLIN.C3*" and "*PWL.NONLIN.C4*"), we conclude the following:

- KHstabilized FRI and FRIPOC FRI methods are not preserving the PWL property to all benchmark Examples ("*PWL.NONLIN.C1*" situations1,2,3, "*PWL.NONLIN.C2*", "*PWL.NONLIN.C3*" and "*PWL.NONLIN.C4*").
- VKK FRI method succeeded with preserving on PWL property in all benchmark examples, except Example "*PWL.NONLIN.C4*", which has appeared with a little bit deviation on the right side.
- VEIN FRI method succeeded with PWL property on benchmark Examples ("*PWL.NONLIN.C1*" situations1,2, "*PWL.NONLIN.C2*" and "*PWL.NONLIN.C3*"), in contrast, the Examples ("*PWL.NONLIN.C1*" situation3 and "*PWL.NONLIN.C4*") have appeared with a little bit deviation in the bottom boundary.
- Incircle FRI method is preserving the PWL property to all benchmark Examples ("*PWL.NONLIN.C1*" situations1,2,3, "*PWL.NONLIN.C2*", "*PWL.NONLIN.C3*" and "*PWL.NONLIN.C4*").

Table 23 presents a summary of the results for the selected FRI methods according to the PWL benchmark examples ("*PWL.NONLIN.C1*" situations1,2,3, "*PWL.NONLIN.C2*", "*PWL.NONLIN.C3*" and "*PWL.NONLIN.C4*") to PWL property, where the plus sign (+) indicates the technique is satisfied with PWL property, while a minus sign (-) shows the method has a little bit deviation in some cases. The cross sign (x) indicates that the method did not preserve the PWL property.

Table 23. Summary of The FRI Methods and Their Conformity to The Benchmark Examples ("PWL.NONLIN.C1" situations1,2,3, "PWL.NONLIN.C2", "PWL.NONLIN.C3" and "PWL.NONLIN.C4").

Examples	Methods					
	KH FRI [1],[9],[13]	KHstab FRI [3]	VKK FRI [2]	FRIPOC FRI [27]	VEIN FRI [23]	Incircle FRI
"PWL.NONLIN.C1" situation1	x	x	+	x	-	+
"PWL.NONLIN.C1" situation2	x	x	+	x	-	+
"PWL.NONLIN.C1" situation3	x	x	+	x	+	+
"PWL.NONLIN.C2"	x	x	+	x	-	+
"PWL.NONLIN.C3"	x	x	+	x	-	+
"PWL.NONLIN.C4"	x	x	-	x	+	+

The Contribution

The dissertation's main contribution is the proposal of a new fuzzy interpolative reasoning method based on the properties of the Incircle triangular fuzzy number. The suggested method is based on the center point of the incircle triangular fuzzy number as a reference point of the fuzzy set. The main sides of the triangular are indicated SD1, SD2, and SD3 (see **Fig. 2**). The tangents length and vertices of the triangle with its Incircle, which denotes PS1, PS2, and PS3, referred to as "fuzziness sides". The proposed Incircle FRI is always producing triangular CNF fuzzy conclusion by holding the same rate of the weights among the observation and the two rule antecedents, and the conclusion and the two corresponding rule consequents with the reference points, and with the "fuzziness sides" (see **Fig. 3**).

The proposed Incircle FRI can also be extended to handle singleton, trapezoidal (see **Fig. 4**), and hexagonal (see **Fig. 5**) membership functions. The proposed Incircle FRI method can also extend to be able to handle fuzzy interpolative reasoning with multiple antecedent variables and multiple fuzzy rules (see **Fig. 6**). Extending the Incircle FRI with general weight calculation and a shift process, the suggested FRI method can also perform extrapolation.

Another important contribution of the dissertation is the proposal of a novel FRI benchmark system. The suggested benchmark is far not ready. The *CNF* (Convex and Normal Fuzzy) and *PWL* (Piecewise Linearity) benchmark examples are just the first step and a methodology to construct a comprehensive FRI (Fuzzy Rule Interpolation) benchmark system, which is built upon the weaknesses of the existing FRI methods, and can also highlight the strengths of the newcomer ones.

3. Scientific Results Summary:

Thesis. I: [39]

I introduced a new method for the fuzzy rule interpolation concept called "Incircle FRI", which is based on the Incircle of a triangular fuzzy number, the Gergonne Point as a "reference point" of the inside circle of triangular fuzzy set, and the fuzziness sides of the triangular. The Incircle FRI conclusion is calculated by holding the same rate of the weights among the observation and the two rule antecedents, and the conclusion and the two corresponding rule consequents with the Gergonne Points (for the reference point of the conclusion), and with the "fuzziness sides" (for left and right fuzziness the shape of the conclusion). The "Incircle FRI" is always generating a triangular CNF conclusion, if the antecedents and the consequents are triangular CNF sets, even if the fuzzy rule-base is sparse. I conclude that the proposed method is a suitable approach to be implemented as an inference system.

Thesis. II: [39]

I introduced an extension of the "Incircle FRI" to be able to handle trapezoidal and hexagonal fuzzy sets, which is by decomposing their membership function shapes into multiple triangulars, and multiple Incircle triangular fuzzy numbers with the Gergonne Points as reference points. I conclude that the extended "Incircle FRI" can generate a trapezoidal, or hexagonal CNF conclusion if the antecedents and the consequents are all trapezoidal, or all hexagonal CNF sets, even if the fuzzy rule-base is sparse. Therefore, the Incircle FRI method is a suitable approach to be implemented as an inference system with trapezoidal and hexagonal fuzzy sets.

Thesis. III:

I introduced an extension of the "Incircle FRI" to be able to handle multiple fuzzy rules having multiple fuzzy antecedents. I used a modification weight estimate and included a shift technique to ensure to interpolate the consequent fuzzy result to be more logical and also to enable the capability for extrapolation. I conclude that the extensions of the "Incircle FRI" always produce CNF conclusion, for all the handled antecedents and consequents configuration of the original method even if the fuzzy rule-base is sparse. Therefore, the Incircle FRI method is a suitable approach to be implemented as an inference system with these extensions.

Thesis. IV: [17], [37]

I introduced the initial benchmark system (set of benchmark examples) for the most important conditions of the FRI concept (CNF and PWL conditions), which is based on analyzing all cases of the core, boundary, and slopes conditions of the antecedents, consequents, and observation fuzzy sets. The KH FRI CNF and PWL Benchmark are suitable for highlighting some problematic points of the

KH FRI and other FRI methods originated from the KH FRI. Therefore, CNF and PWL Benchmarks are suitable for evaluating and comparing FRI methods, where the KH FRI is not satisfied with CNF and PWL properties.

Author's Publication

*[11]. M. Alzubi, Z. C. Johanyak, and S. Kovacs, "**Fuzzy Rule Interpolation Methods and FRI Toolbox**", **Journal Scopus Indexed [Q3]**.*

*[12]. M. Alzubi and S. Kovacs, "**Investigating the piecewise linearity and benchmark related to KOCZY-HIROTA fuzzy linear interpolation**", **Journal Scopus Indexed [Q3]**.*

*[24]. M. Alzubi and S. Kovacs, "**Some Considerations and a Benchmark Related to the CNF Property of the Koczy-Hirota Fuzzy Rule Interpolation**", **Journal Scopus Indexed [Q2]**.*

*[25]. Maen Alzubi, Mohammad Almseidin, Szilveszter Kovacs, and Mohd Aaqib Lone, "**Fuzzy Rule Interpolation Toolbox for the GNU Open-Source OCTAVE**", **Conference [IEEE]**.*

*[26]. M. Alzubi and S. Kovacs, "**Interpolative Fuzzy Reasoning Method Based on the Incircle of a Generalized Triangular Fuzzy Number**", **Journal Scopus Indexed [Q1]**.*

REFERENCES:

[1] L. Kóczy and K. Hirota, "*Approximate reasoning by linear rule interpolation and general approximation*", International Journal of Approximate Reasoning, Vol. 9, No. 3, 1993, pp. 197–225. https://doi.org/10.1016/0888-613X(93)90010-B.

[2] G. Vass, L. Kalmar, and L. Kóczy, "*Extension of the fuzzy rule interpolation method*", in Proc. Int. Conf. Fuzzy Sets Theory Applications, 1992, pp. 1–6.

[3] Tikk, D., Joó, I., Kóczy, L., Várlaki, P., Moser, B., & Gedeon, T. D., "*Stability of interpolative fuzzy KH controllers*", Fuzzy Sets and Systems, Vol. 125, No. 1, 2002, pp. 105–119. https://doi.org/10.1016/S0165-0114(00)00104-4.

[4] D. Tikk and P. Baranyi, "*Comprehensive analysis of a new fuzzy rule interpolation method*", IEEE Transactions on Fuzzy Systems, Vol. 8, No. 3, 2000, pp. 281–296. doi: 10.1109/91.855917.

[5] L. T. Kóczy, K. Hirota, and T. D. Gedeon, "*Fuzzy rule interpolation by the conservation of relative fuzziness*", JACIII, Vol. 4, No. 1, 2000, pp. 95–101. doi: 10.20965/jaciii.2000.p0095.

[6] K. W. Wong, T. Gedeon, and D. Tikk, "*An improved multidimensional α-cut based fuzzy interpolation technique*", in Proc. Int. Conf Artificial Intelligence in Science and Technology (AISAT), Hobart, Australia, 2000, pp. 29-32. http://researchrepository.murdoch.edu.au/id/eprint/1034.

[7] Z. Huang and Q. Shen, "*Fuzzy interpolative reasoning via scale and move transformations*", IEEE Transactions on Fuzzy Systems, vol. 14, no. 2, 2006, pp. 340–359. DOI: 10.1109/TFUZZ.2005.859324.

[8] Z. C. Johanyák, D. Tikk, S. Kovács, and K. W. Wong, "*Fuzzy rule interpolation MATLAB toolbox-FRI toolbox*", in Fuzzy Systems, 2006 IEEE International Conference on. IEEE, 2006, pp. 351–357. doi: 10.1109/FUZZY.2006.1681736.

[9] L. Kóczy and K. Hirota, "*Interpolative reasoning with insufficient evidence in sparse fuzzy rule bases*", Information Sciences, vol. 71, no. 1-2, 1993, pp. 169–201. https://doi.org/10.1016/0020-0255(93)90070-3

[10] The FRI Toolbox is available at: *http://fri.uni-miskolc.hu*.

[11] M. Alzubi, Z. C. Johanyak, and S. Kovacs, "*Fuzzy rule interpolation methods and FRI toolbox*", Journal of Theoretical and Applied Information Technology, vol. 96, pp. 7227–7244, 2018.

[12] M. Alzubi and S. Kovacs, "*Investigating the piecewise linearity and benchmark related to Koczy-Hirota fuzzy linear interpolation*", arXiv preprint arXiv:1907.01047, 2019.

[13] L. T. Koczy, K. Hirota, "*Size reduction by interpolation in fuzzy rule bases*", IEEE Transactions on Systems, Man, and Cybernetics, Part B (Cybernetics), vol. 27, no. 1, pp. 14–25, 1997. doi:10.1109/3477.552182.

[14] L. T. Koczy, S. Kovacs, *"the convexity and piecewise linearity of the fuzzy conclusion generated by linear fuzzy rule interpolation"*, J. BUSEFAL, vol. 60, pp. 23–29, 1994. URL https://projects.listic.univ-smb.fr/busefal/papers/60.zip/60_04.pdf

[15] W.-H. Hsiao, S.-M. Chen, C.-H. Lee, *"A new interpolative reasoning method in sparse rule-based systems"*, Fuzzy Sets and Systems, vol. 93, no. 1, pp. 17–22, 1998. doi:10.1016/S0165-0114(96)00190-X.

[16] L.T. Koczy, S. Kovacs, *"Linearity and the CNF property in linear fuzzy rule interpolation"*, in Fuzzy Systems, 1994. IEEE World Congress on Computational Intelligence., Proceedings of the Third IEEE Conference on (IEEE, 1994), pp. 870–875.

[17] L. Koczy, *"Algorithmic aspects of fuzzy control"*, International Journal of approximate reasoning, vol. 12, no. 3-4, 1995, pp. 159–219.

[18] Chang, Y.C.; Chen, S.M.; Liau, C.J. *"Fuzzy interpolative reasoning for sparse fuzzy-rule-based systems based on the areas of fuzzy sets"*. IEEE Transactions on Fuzzy Systems 2008, 16, 1285–1301.

[19] Huang, D.M.; Tsang, E.; Yeung, D. *"A fuzzy interpolative reasoning method"*. Proceedings of 2004 International Conference on Machine Learning and Cybernetics (IEEE Cat. No. 04EX826). IEEE, 2004, Vol. 3, pp. 1826–1830.

[20] Zhou, W.J.; Maskell, D.L.; Quek, C. *"Fuzzy interpolation and extrapolation using shift ratio and overall weight measurement based on areas of fuzzy sets"*. 2013 13th UK Workshop on Computational Intelligence (UKCI). IEEE, 2013, pp. 46–53.

[21] Chen, S.M.; Ko, Y.K. *"Fuzzy Interpolative Reasoning for Sparse Fuzzy Rule-Based Systems Based on α-cuts and Transformations Techniques"*. IEEE Transactions on Fuzzy Systems 2008, 16, 1626–1648.

[22] Chang, Y.C.; Chen, S.M.; Liau, C.J. "A new fuzzy interpolative reasoning method based on the areas of fuzzy sets". 2007 IEEE International Conference on Systems, Man and Cybernetics. IEEE, 2007, pp. 320–325.

[23] Johanyák, C.Z., *"Vague Environment Based Set Interpolation"*. A GAMF KÖZLEMÉNYEI, p.33, 2007.

[24] M. Alzubi and S. Kovacs, *"Some Considerations and a Benchmark Related to the CNF Property of the Koczy-Hirota Fuzzy Rule Interpolation"*, International Journal on Advanced Science, Engineering and Information Technology", Vol. 9 (2019) No. 5, pages: 1761-1771, DOI:10.18517/ijaseit.9.5.8356.

[25] M. Alzubi, M. Almseidin, S. Kovacs, and Mohd Aaqib Lone, *"Fuzzy Rule Interpolation Toolbox for the GNU Open-Source OCTAVE"*, International Conference on Emerging eLearning Technologies and Application ", arXiv preprint arXiv:1912.04999 (2019).

[26] M. Alzubi and S. Kovacs, *"Interpolative Fuzzy Reasoning Method Based on the Incircle of a Generalized Triangular Fuzzy Number"*, "Journal of Intelligent & Fuzzy Systems", DOI: 10.3233/JIFS-191660, vol. Pre-press, no. Pre-press, pp. 1-21, (2020).

[27] Z. C. Johanyák and S. Kovács, "*Fuzzy rule interpolation based on polar cuts*", in Computational Intelligence, Theory and Applications. Springer, 2006, pp. 499–511.