



UNIVERSITY OF MISKOLC  
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**Three-field dual-mixed variational formulation  
and  $hp$  finite element model for elastodynamic  
analysis of axisymmetric shells**

Booklet of PhD Theses

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Miskolc-Egyetemváros

2012

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# 1 Preliminary

## 1.1 Displacement-based finite element methods

The most common approach to construct finite element models for linear elasticity problems is to consider the *pure displacement formulation*, where the displacement is the only variable and the solution is based on the global minimization of the total potential energy functional. In fact, this leads to the principle of virtual work ensuring the satisfaction of the translational equilibrium equation and the traction boundary condition in a weak sense, while the kinematic equation and the displacement boundary condition are enforced exactly. The simplest classical finite elements, such as the beam, the triangular and the rectangular element, were first published in the 1950's. These elements use linear piecewise polynomials to approximate the solution, and achieve increased accuracy with mesh refinement. This philosophy of using low-order polynomials over successively finer meshes called *h*-type approximation technique.

The widely used standard displacement finite element technique has some significant disadvantages. Stability problems can be expected at nearly incompressible materials, when the Poisson ratio is close to the incompressibility limit of 0.5, i.e., when one of the Lamé constants tends to infinity. In this case the displacements can be inaccurately computed and even worse values can be obtained for the sum of the normal stresses. This is the well-known *incompressibility locking* effect. Similar locking problems appear in the finite element computations of thin plates and shells, see Subsection 1.3 for details. Furthermore, the conventional finite element models provide slow convergences and low accuracy in the evaluation of the stress field, which is often more important in many engineering applications than the knowledge of displacements. Namely, the stresses are obtained through the numerical differentiation of the approximated displacements, thus leading to this decrease in accuracy.

Experiments of the research group conducted by Barna Szabó in the mid-1970's indicated that an alternative strategy might hold great

promise. Their idea was to keep the coarse mesh fixed, and the convergence is achieved solely by increasing the polynomial degree  $p$  of the approximated displacements. This finite element technique is called as  $p$ -version to be distinguished from the classical  $h$ -version method, where  $p$  is fixed and the mesh is refined. The computational results obtained for elasticity problems indicated that the new philosophy was always competitive with, and often out-performed, the conventional  $h$ -version. It was proven that the convergence rate of the  $p$ -version often is double than that is possible with the  $h$ -version, and exponential for the displacement solution. Although the higher-order displacement elements give optimal accuracy for the displacements, the stresses recovered by postprocessing from the displacement field are usually not free from the incompressibility locking.

## 1.2 Mixed methods

One of the strategies to overcome the incompressibility locking effect for low-order  $h$ -version finite elements is to apply mixed<sup>1</sup> methods. One of them is the Herrmann-type primal<sup>2</sup>-mixed variational principle, where the incompressibility constraint equation is satisfied only approximately, in a weak sense. This method is known as the displacement-pressure formulation for nearly incompressible materials and as the velocity-pressure formulation for the Stokes problem in fluid mechanics. The theoretical and numerical results showed that these primal-mixed  $h$  and  $p$  elements have excellent incompressibility locking-free approximation properties not only for the displacements but also for the stresses.

An alternative way to avoid the incompressibility locking effect for low-order  $h$ -type finite elements is to apply dual<sup>3</sup>-mixed variational

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<sup>1</sup>‘mixed’ indicates that the number of the independent fields is at least two in the variational principle.

<sup>2</sup>‘primal’ means that the displacement is the primary variable and its gradient appears in the formulation.

<sup>3</sup>‘dual’ implies that the primary unknown is the stress field and its divergence appears in the formulation.

principles in the framework of linear elasticity. These methods are less, or not sensitive to the material properties and deliver better convergence rates and higher accuracy for the stresses, than pure displacement formulations and strain energy-based primal-mixed methods mentioned previously. The weak solution of the classical dual formulation is based on the global maximization of the total complementary energy functional in terms of the stress field. In fact, this is equivalent to the principle of complementary virtual work, satisfying the strain compatibility equation and the displacement boundary condition in a weak sense. The traction boundary condition, as well as the translational and rotational equilibrium equation are enforced in the strong sense using the second-order stress function tensor. It is the so-called *equilibrium method*. The related finite element schemes require, however,  $C^1$  continuous approximation of the second-order stress functions. This requirement, which is primarily due to the *a priori* satisfaction of the symmetry condition for the stress tensor, makes it rather difficult and complicated to establish a numerically efficient and well manageable equilibrium method and stress elements for general problems.

One of the possibilities to overcome the difficulties mentioned in connection with the development of numerically efficient stress-based formulation is to incorporate the symmetry condition for the stress tensor into the total complementary energy functional using the rotations as Lagrangian multipliers. Applying this two-field dual-mixed variational principle named after Fraeijns de Veubeke, the translational equilibrium equation has to be satisfied *a priori* with the first-order stress function tensor. The most important advantage of the finite element methods based on the Fraeijns de Veubeke principle is that the approximation of the first-order stress function tensor requires only  $C^0$  continuity at the interelement boundaries.

Other alternative strategy to construct complementary energy-based finite element models with optimal convergence rates for stresses and displacements, is to satisfy the translational equilibrium equation weakly using the displacement field as Lagrangian multiplier and letting the stress field be *a priori* symmetric. Thus the classical two-field varia-

tional principle proposed originally by Ernst David Hellinger and Eric Reissner are obtained, in which the stress tensor and the displacement vector are simultaneously approximated as independent variables. Since in the two-field primal-mixed Hellinger–Reissner functional the gradient operator is applied to the displacement vector, the displacements have to be  $C^0$  continuous, while the normal components of the stress tensor are discontinuous across the element interfaces. However, it has the major shortcoming that the order of convergence for the stresses is lower by one than that is possible with the use of finite element models based on Fraeijs de Veubeke’s or classical dual principles. To avoid this, the divergence operator is shifted from the displacement vector to the stress tensor obtaining the two-field dual-mixed Hellinger–Reissner functional. In this case the normal components of the stress tensor have to be  $C^0$  continuous and the displacements are discontinuous at the interelement boundaries. The simultaneous strong enforcement of the symmetry and the continuity condition causes, however, substantial difficulties in finite element modeling.

The lack of simple stable and efficient dual-mixed three-dimensional elements for the two-field dual-mixed Hellinger–Reissner functional has led to the construction of modified three-field Hellinger–Reissner-type variational principles in which the symmetry of the stresses is fulfilled in a weak sense via the rotations as Lagrangian multipliers. This approach has numerous advantages over the traditional displacement finite element models. It provides more accurate approximation and higher convergence rate for the stress field which is usually the variable of primary interest in engineering practice. Furthermore, it helps to overcome several locking phenomena.

### 1.3 Numerical locking effects in shell modeling based on dimensional reduction procedure

The classical shell theories are usually derived by applying the so-called *dimensional reduction procedure*, where the original three-dimensional shell problem is replaced with an approximate two-dimensional one.

The standard dimensionally reduced shell models and the related low-order finite elements suffer from several numerical problems. The basic source of these numerical difficulties arises from the small value of the thickness. This phenomenon is, again, a *numerical locking* effect.

There are several types of numerical locking in dimensionally reduced shell modeling. The transverse *shear locking* appears in bending dominated cases when the bending energy is restrained and nearly all the strain energy is stored in transverse shear energy terms for small thicknesses. This is partly caused by the fact that the finite element solution of the Naghdi-model is forced to satisfy the Kirchhoff–Love hypothesis as the value of the shell thickness tends to zero. The even more severe *membrane locking* occurs when the bending energy is restrained and the total strain energy is stored in membrane energy terms for small thickness values. The numerical locking in *boundary layers* is a more hidden effect, and it can also cause significant errors in local mechanical quantities such as stress maxima. One of the effective strategies to overcome this singular behavior is the use of local mesh refinement near the boundary.

There are several attempts to circumvent transverse shear locking. One of them is to modify the principle of virtual work to enforce the Kirchhoff–Love constraint in a weak sense. For low-order  $h$ -type analyses, such *reduced constraint methods* seem to be promising to avoid this locking effect.

Another way to overcome this kind of locking is the application of the *reduced* or *selective integrations*. Nevertheless, the use of these methods is often accompanied by spurious zero energy modes for certain boundary conditions and mesh-layouts. Membrane locking can be treated similarly as shear locking in the case of bending dominated shell problems. However, the shell elements developed in such a way do not perform very well up to now, when they are applied to membrane dominated cases.

One of the most reliable locking removal techniques is based on the *mixed interpolation of tensorial components* (MITC), where the interpolations of the strain tensor components are appropriately chosen for



the related displacement interpolations and tied to the displacements at special points. The theoretical basis of this method was originally introduced by Klaus-Jürgen Bathe and Eduardo Dvorkin. A large number of papers have dealt with the numerical efficiency of these shell finite elements.

Another method to construct shear locking-free plate and shell bending finite elements is the so-called *discrete Kirchhoff technique* (DKT). The independent variables of the element formulations based on this approach are usually the transverse deflection and two infinitesimal ‘rotation’ components, which require only  $C^0$  continuous approximations at the interelement boundaries. Furthermore the transverse shear energy terms are neglected, i.e., the bending strain energy is only considered, as well as the Kirchhoff–Love constraints are imposed in a discrete way on the element sides. The numerical behavior of the DKT elements of various shapes was studied and discussed mainly by Jean-Louis Batoz *et al.*

Additional efficient strategy for avoiding locking problems in the displacement-based finite element formulation is to use high-order methods and  $p$ -version finite elements. These  $p$  elements are verified to be locking-free in the energy norm computations for general shells, but the numerical results obtained for stresses are not exempt from locking.

Another solution for locking problems is the application of stress-based variational principles. The design of this kind of  $hp$ -version shell elements is complicated and more demanding task than the construction of classical displacement-based shell finite element models. A general shell model in terms of symmetric stresses had been constructed without the use of the classical kinematical hypotheses by Imre Kozák. Subsequently, a complementary energy-based cylindrical shell model, using the second-order stress function tensor, was also derived. Applying the two-field dual-mixed variational principle of Fraeijns de Veubeke, locking-free  $hp$ -version plate and shell elements were presented for one- and two-dimensional elastostatic problems.

The two-field primal- and dual-mixed Hellinger–Reissner variational principle was applied to the elastostatic problems of plates by Douglas Arnold *et al.* The application of these kinds of variational principles and the pure dual formulation was also extended to the vibration analysis of plates and axisymmetric shells by Wolf Altman *et al.* and Behrouz Tabarrok *et al.* Using the three-field primal-mixed variational principle of Hellinger–Reissner-type,  $h$ -version plate and shell elements were developed for linear and nonlinear problems mainly by Satya Atluri *et al.* Nevertheless, these plate and shell finite element models are based on the Kirchhoff–Love hypothesis and modified constitutive equations. To the author’s knowledge, dimensionally reduced axisymmetric shell model based on the three-field dual-mixed variational principle of Hellinger–Reissner-type, without the classical hypotheses, cannot be found in the current literature.

Beside the use of the two-dimensional shell finite element models, the application of the so-called *degenerated elements* derived from the three-dimensional shell theories are also widespread. These shell elements are also based on the Naghdi-type kinematics, but the integrations with respect to the thickness coordinate are carried out numerically. Nevertheless, not only the numerical problems that occur in the dimensionally reduced finite element shell models but also additional numerical locking effects (*curvature locking, dilatation locking, thickness locking, trapezoidal locking, incompressibility locking*) appear in the case of the degenerated shell elements.

## 2 Objectives

The developments presented in the dissertation have been motivated by (i) the lack of general locking-free shell finite element that is robust and reliable in both bending- and membrane-dominated situations, for both  $h$ - and  $p$ -extensions, and give accurate and reliable numerical results not only for the displacements but also for the stresses, and (ii) the limited applicability of the principle of complementary virtual work to time-dependent problems.

The main goal of the research work was to develop and present a new dimensionally reduced axisymmetric shell model based on a three-field dual-mixed variational principle for linear elastodynamic problems. The functional of this principle for time-dependent analyses can be written in the form

$$\mathcal{F}(\sigma^{kl}, u_k, \phi^s) = \int_{t_0}^{t_1} \left( \mathcal{H} \mathcal{R} - \widehat{\mathcal{K}} \right) dt, \quad (1)$$

where

$$\widehat{\mathcal{K}}(\dot{u}_k) = \int_V \widehat{\mathcal{T}} dV = \frac{1}{2} \int_V \rho \dot{u}^k \dot{u}_k dV \quad (2)$$

is the complementary kinetic energy of the elastic body and

$$\begin{aligned} \mathcal{H} \mathcal{R}(\sigma^{kl}, u_k, \phi^s) &= - \int_V \widehat{\mathcal{U}} dV + \int_{S_u} \tilde{u}_k \sigma^{kl} n_\ell dS \\ &\quad - \int_V [u_k (\sigma^{kl}{}_{;l} + b^k) - \phi^s \epsilon_{k\ell s} \sigma^{kl}] dV \end{aligned} \quad (3)$$

is the three-field dual-mixed Hellinger–Reissner-type functional of elastostatics. Here  $V$  denotes the volume of the body, the surface  $S = S_p \cup S_u$ , with  $S_p \cap S_u = \emptyset$ , is the boundary of  $V$ ,  $\epsilon_{k\ell s}$  is the covariant permutation tensor and  $\tilde{u}_k$  is the displacement vector prescribed on the surface part  $S_u$  with outward unit normal  $n_\ell$ , as well as  $b^p$  and  $\rho$  stand, respectively, for the density of the body forces and the material, and  $t \in [t_0, t_1]$  defines a closed time interval ( $t_0$  and  $t_1$  are two arbitrary instants of time). The fundamental variables of functional (1) are the not *a priori* symmetric stress tensor  $\sigma^{kl}$ , the displacements  $u_p$  and the rotations  $\phi^s$ . The complementary strain energy density function  $\widehat{\mathcal{U}}$  is defined by

$$\widehat{\mathcal{U}}(\sigma^{kl}) = \frac{1}{2} \sigma^{pq} \varepsilon_{pq}(\sigma^{kl}). \quad (4)$$

For linearly elastic materials the symmetric strain tensor  $\varepsilon_{pq}$  can be obtained from the inverse stress-strain relations (Hooke's law)

$$\varepsilon_{pq} = C_{pqkl} \sigma^{kl} \quad \text{in } V, \quad (5)$$

where the fourth-order tensor  $C_{pqkl}$  with symmetry properties  $C_{pqkl} = C_{pqlk} = C_{k\ell pq}$  is the elastic compliance tensor. According to the variational principle, the solution of the linear elastodynamic problem can be characterized as the stationary point of functional (1) over the space of all vector fields  $u_p$ ,  $\phi^s$  and all *a priori* non-symmetric stress fields  $\sigma^{k\ell}$  satisfying the stress boundary conditions

$$f^k = \sigma^{k\ell} n_\ell \quad \text{on } S_p, \quad (6)$$

where  $f^k$  are prescribed surface tractions on  $S_p$  with outward unit normal  $n_\ell$ . The initial conditions to (1) are

$$u_k(t_0) = {}^0u_k, \quad v_k(t_0) = {}^0\dot{u}_k = {}^0v_k \quad \text{in } V. \quad (7)$$

Considering the above dual-mixed variational formulation and principle, the aims of the dissertation have been (1) the derivation of the special form of the three-field dual-mixed functional to thin shells of revolution, (2) the development of a new dimensionally reduced axisymmetric shell model without the application of the classical kinematical hypotheses regarding the deformation of the normal to the shell mid-surface, (3) the derivation of the Euler–Lagrange equations and natural boundary conditions, as well as the fundamental differential equation system of the new shell model in terms of non-symmetric stresses, displacements and rotations, (4) the development of a new dual-mixed finite element formulation and the related *hp*-version mixed shell finite element models for elastodynamic problems of thin cylindrical shells, (5) construction of new algorithms and finite element codes for numerical investigation of cylindrical shell problems, and (6) comparison of the new dual-mixed shell model and elements to other shell finite element models through their numerical performances.

## 3 Theses

### Thesis 1

I have extended the dual-mixed Hellinger–Reissner-type variational formulation to linear elastodynamic problems. The independent fields of the new four-field variational principle are the displacement vector, the non-symmetric stress tensor, the skew-symmetric rotation tensor and the impulse vector. Eliminating the impulse field from the four-field functional, I have derived a three-field dual-mixed variational principle for elastodynamic analyses of three-dimensional time-dependent problems. This three-field functional served as a basis for further developments presented in the dissertation and in the subsequent theses.

► Related publications: [8], [9], [12], [13], [15].

### Thesis 2

I have derived the special form of the three-field dual-mixed functional of Thesis 1 for thin shells of revolution, applying the differential geometric description of surfaces and shells. I have developed a new dimensionally reduced shell model for linear elastodynamic problems of axisymmetric shells. The fundamental (independent) variables of the shell model developed are the displacement vector, the non-symmetric stress tensor and the skew-symmetric rotation tensor. The shell model developed does not rely on the classical kinematical hypotheses regarding the deformation of the normal to the shell middle surface and, thus, uses unmodified three-dimensional constitutive equations.

► Related publications: [1], [9], [12], [13], [15].

## Thesis 3

Starting from the special form of the three-field dual-mixed variational principle for axisymmetric shells of Thesis 2, I have derived the Euler–Lagrange equations and the natural boundary conditions of the new dimensionally reduced shell model, assuming axisymmetric loads and homogeneous isotropic materials. The variational equations have been formulated in terms of one-dimensional variables defined on the middle surface of the shell and consist of the special forms of the translational and rotational equations of motion, the kinematic equations and the displacement boundary conditions. I have derived the fundamental differential equation system for the independent variables of the axisymmetric shell model.

► Related publications: [2], [4–6], [10], [16], [17], [19].

## Thesis 4

I have developed a new dual-mixed  $hp$  finite element model for axisymmetric elastodynamic problems of thin cylindrical shells. This is based on a modified version of the three-field dual-mixed variational principle. The modification has resulted in a two-field formulation in which the symmetry conditions are satisfied in an integral average sense through the related Euler–Lagrange equations. The polynomial interpolation spaces applied assume  $C^0$ -continuous approximation for the normal stress components and discontinuous approximation for the displacements at the element interfaces. The computational performance of the new dual-mixed  $hp$  finite element model has been tested and demonstrated for static and dynamic problems of cylindrical shells, for both  $h$ - and  $p$ -extensions. The numerical results have been compared to the solutions of displacement-based models and analytical solutions. I have proven through these computations and comparisons that the shell model and the related  $hp$ -version shell finite elements developed in this dissertation are robust and reliable, i.e., they are free from incompressibility- and shear locking-effects, not only for the displacements, but also for the stress computations.

► Related publications: [3], [7–9], [11–15], [18], [19].

## 4 Author's publications

The following publications were made in the topic of the dissertation:

### Articles in journals:

- [1] Tóth, B. A Hellinger–Reissner-féle variációs elv alakja forgáshéjakra. (The Hellinger–Reissner variational principle for axisymmetric shells). *GÉP*, **LVIII**(5-6):48–53, 2007.
- [2] Tóth, B. Axisymmetric shell model using a three-field dual-mixed variational principle. *Journal of Computational and Applied Mechanics*, Accepted: November 12, 2008. unpublished full article.
- [3] Tóth, B. Dual-mixed *hp* finite element model for elastic cylindrical shells. *ZAMM, Journal of Applied Mathematics and Mechanics*, **92**:236–252, 2012. (*IF*=0.948).

### Conference papers:

- [4] Tóth, B. Application of the Hellinger–Reissner variational principle to shells of revolution. In *Proceedings of microCAD 2007 International Scientific Conference*, pages 47–51, University of Miskolc, Hungary, Section F. Applied Mechanics, March 22–23, 2007.
- [5] Tóth, B. The Hellinger–Reissner variational principle for axisymmetric shells. In *Proceedings of Ph.D. Students' Forum*, pages 158–166, University of Miskolc, Hungary, November 13, 2007.
- [6] Tóth, B. Axisymmetric shell model using the three-field Hellinger–Reissner's variational principle. In *Proceedings of microCAD 2008 International Scientific Conference*, pages 47–51, University of Miskolc, Hungary, Section F. Applied Mechanics, March 20–21, 2008.

- [7] Tóth, B. Dual-mixed finite element model for elastic cylindrical shells. In *Proceedings of XXIII. microCAD International Scientific Conference*, pages 51–60, University of Miskolc, Hungary, Section F. Applied Mechanics, March 19–20, 2009.
- [8] Tóth, B. Dual-mixed Hellinger–Reissner functional for elastodynamics. In *Proceedings of XXIV. microCAD International Scientific Conference*, pages 73–77, University of Miskolc, Hungary, Section G. Applied Mechanics, March 18–20, 2010.
- [9] Tóth, B. Vibration analysis of cylindrical shells based on a dual-mixed finite element formulation. In *Proceedings of XXV. microCAD International Scientific Conference*, pages 49–56, University of Miskolc, Hungary, Section E. Applied Mechanics, March 31 – April 1, 2011.

## Conference presentations:

- [10] Tóth, B. A hárommezős Hellinger–Reissner-féle variációs elv alkalmazása forgáshéjakra (The three-field Hellinger–Reissner variational principle for shells of revolution). *X. Magyar Mechanikai Konferencia*, University of Miskolc, Hungary, August 27–29, 2007.
- [11] Tóth, B. A duál Hellinger–Reissner-féle variációs elven alapuló *hp*-verziós végelem-modell körhengerhéjakra (An *hp* finite element model based on dual-mixed Hellinger–Reissner variational principle for cylindrical shells). *XI. Magyar Mechanikai Konferencia*, University of Miskolc, Hungary, August 29–31, 2011.
- [12] Tóth, B. A hárommezős Hellinger–Reissner-féle variációs elv alkalmazása forgáshéjak elasztodinamikai feladataira (Three-field Hellinger–Reissner variational principle for elastodynamic problems of axisymmetric shells). *XI. Magyar Mechanikai Konferencia*, University of Miskolc, Hungary, August 29–31, 2011.



- [13] Tóth, B. Dual-mixed  $hp$  finite element model for elastodynamic problems of cylindrical shells. *38th Solid Mechanics Conference*, Warsaw, Poland, August 27–31, 2012.
- [14] Tóth, B. and Bertóti, E. Dual-mixed finite element method for elastic cylindrical shells. *37th Solid Mechanics Conference*, Warsaw, Poland, September 6–10, 2010.
- [15] Tóth, B. and Bertóti, E. A three-field dual-mixed  $hp$  finite element model for cylindrical shells. *HOFEIM, Workshop on Higher Order Finite Element and Isogeometric Methods*, Cracow, Poland, June 27–29, 2011.

## Talks:

- [16] Tóth, B. Application of a three-field dual-mixed variational principle to axisymmetric shells. *Otto von Guericke University, Institute of Mechanics*, Magdeburg, Germany, December 17, 2007.
- [17] Tóth, B. A Hellinger–Reissner-féle variációs elv alkalmazása forgáshéjakra (Application of the Hellinger–Reissner variational principle to axisymmetric shells). *Doktoranduszok Szemináriuma (Ph.D. Students’ Seminar)*, Miskolci Egyetem (University of Miskolc, Hungary), November 19, 2007.
- [18] Tóth, B. Three-field dual-mixed finite element model for cylindrical shells. *Otto von Guericke University, Institute of Mechanics*, Magdeburg, Germany, December 18, 2008.
- [19] Tóth, B. A hárommezős duál Hellinger–Reissner-féle variációs elv alkalmazása körhengerhéjra, (Application of the three-field dual-mixed Hellinger–Reissner variational principle to cylindrical shells). *Doktoranduszok Szemináriuma (Ph.D. Students’ Seminar)*, Miskolci Egyetem (University of Miskolc, Hungary), April 6, 2009.